Poisson Statistics:

What is the Poisson statistics? How do we derive it from scratch?

Examples of poissonian distribution can be found everywhere. The number of raindrops in a specific area in a given amount of time. The number of nuclear decay in a given mass of uranium in a given time. This distribution can be defined as the number of times a random event occurs within a certain window (time or otherwise).

The Poisonnian distribution can be derived from scratch, provided you know the following two things:

1. Combinatorial ${}^nC_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)...(n-r+1)}{r!}$
2. $(1+x) = e^x$ when x is vanishingly small.

$(1+x) = e^x$?

This is only true when x is vanishing small. We can get to this from the taylor series expansion of e^x.

$$e^x = 1 + x + \tfrac{1}{2}x^2 + ...$$
$$\Rightarrow e^x = 1 + x \quad when\ x \to 0$$

Of cos, you can reexpress $(1+x)$ as any number of functions where the first 2 terms of their taylor series expansion is $(1+x)$, but rewriting it as e^x has its advantages. One of the more useful advantages is that $(1+x)^n$ can now be expressed as e^{nx}. And this is the one that we will make use of.

Let's consider a real life situation. Assume a light rain and that you have a bucket. Suppose the average number of raindrops collected over a time T is λ.

We can find out the probability of collecting N drops in a given time window T by slicing the time interval into a bunch of very small intervals (let's say n of them; where n tends to infinity). Because each of the intervals is very short, it is unlikely that a raindrop will fall within each interval. More importantly, it is extremely unlikely that more than 1 raindrop will fall within each time interval. The probability of having a raindrop in a given time interval is $p = \frac{\lambda}{n}$.

To get no raindrop in the whole time window, we cannot have any raindrop in any of the time windows. This means that we can write $P(0\ raindrop)$ or $P(0)$ as

$$P(0) = \left(1 - p\right)^n . \tag{1}$$

Using $e^x = 1 + x \quad when\ x \to 0$, we get

$$P(0) = e^{-np} = e^{-\lambda} . \tag{2}$$

To get N raindrops in the whole time window, we must have one rain drop occurring in N time intervals. The permutation is combinatory, so we get:

$$P(N) = p^N \left(1 - p\right)^{n-N} \ {}^nC_N = \left(1 - p\right)^n \left(\frac{p}{1-p}\right)^N \frac{n(n-1)...(n-N+1)}{N!} . \tag{3}$$

Remember that $np = \lambda$ and we have $n \to \infty$ and $p \to 0$, so

$$\left(\frac{p}{1-p}\right)^N \approx p^N , \tag{4}$$

$$\frac{n(n-1)...(n-N+1)}{N!} \approx \frac{n^N}{N!} . \tag{5}$$

So,

$$P(N) = e^{-np} \frac{(np)^N}{N!} = e^{-\lambda} \frac{\lambda^N}{N!} \tag{6}$$

We can check and see that:

$$\sum_{N=0}^{\infty} P(N) = \sum_{N=0}^{\infty} e^{-\lambda} \frac{\lambda^N}{N!} = e^{-\lambda} e^{\lambda} = 1 \tag{7}$$

The mean number of raindrop in the time window can be calculated from:

$$E(N) = \sum_{N=0}^{\infty} NP(N) = \sum_{N=0}^{\infty} Ne^{-\lambda} \frac{\lambda^N}{N!} = \sum_{N=1}^{\infty} Ne^{-\lambda} \frac{\lambda^N}{N!} = \lambda \sum_{N=0}^{\infty} e^{-\lambda} \frac{\lambda^N}{N!} = \lambda \tag{8}$$

Not surprising….

$$E(N^2) = \lambda^2 + \lambda \tag{9}$$

The standard dev is therefore:

$$stdev(N) = \sqrt{E(N^2) - E(N)^2} = \sqrt{\lambda} \tag{10}$$

<u>Suppose we decide to count photons from a weak light source. Why should we expect the standard deviation of photon count to equal to the mean photon count?</u>

The stdev of poissonian statistics is equal to square root of its mean. Since, counting photons is equivalent to counting raindrops, the stdev of photon counts must equal the square root of the mean.

<u>Are there situations where the photon flux does not obey poissonian statistics?</u>

There certainly are. Here's a very simple example. Consider a situation where you are looking for fluorescence photons that are emitted from a single molecule. If the molecule takes some amount of time to recover between each fluorescence emission event, the collected photon count will no longer be Poissonian. In our Poissonian statistics derivation, we placed no restrictions on how close any two given events can be to each other; while the concept of a recovery time here clearly implies a need for some restrictions.

A very good research example of non-Poissonian photon statistics can be found in fluorescence correlation spectroscopy (google it if you are interested.).

In this present situation, the recovery time is exponentially distributed. Can you work out the probability distribution function?

<u>Suppose we use a photodetector to count photons. If the quantum efficency (the probability of a photon generating the flow of an electron) of the photodetector is less than unity, will the statistics of the electron count mirror that of the photons'?</u>

It is easy to see that if the quantum efficiency is unity, the statistics of the electron count from the detector will be identical to that of the photon flux. However, when the quantum efficiency is less than unity, the two statistics can be pretty different.

Here's an example. Consider a source where photons are emitted at a regular interval. If the detector is not 100% efficient, the time difference between consecutive electrons will not be uniform. Which implies a different statistical distribution.

In fact, if we are collecting the electrons over a very long time interval and the quantum efficiency is very low, the electron statistics will approach Poissionian. (Can you see why? Hint: look at the Poissonian statistics derivation).

I like to look at Poissonian statistics as a sort of 'ground state' statistical distribution. It is very easy to 'degrade' other forms of statistics into a Poissonian one, by simply having an inefficient detection system, long enough data collection time, etc….

Having said that, let’s consider the case where the photon flux is poissonian. In this case, no matter what the quantum efficiency of the photodetecotr is, the electron flow will be Poissonian too. The mean will be the product of the mean for the photon flux and the quantum efficency. We can see this easily by going back to the rainfall and bucket picture. The photon count in a given time window is equivalent to counting raindrops in the bucket for a given time window. The inefficient photo detector selects a fraction of those photons for conversion into electrons. This is equivalent to having a smaller bucket in the original bucket, counting the raindrop in this smaller bucket is equivalent to counting the converted electron flow. So at the end of the day, counting electron flow is equivalent to having a smaller bucket sitting out in the rain. The electron flow must therefore be Poissionian.

What is the frequency profile of a time trace for photon flux (or any other phenomenon) that obeys a Poissonian distribution?

The frequency profile will be flat over all frequencies. There’s an easy way to see this. If the frequency is not flat, there will be a frequency component that is greater than any other. This implies that the time trace will show a certain amount of periodicity (specifically at the abovementioned frequency). But, Poissonian distribution implies that there is no correlation between the arrival time of each photon, so there cannot be any periodicity associated with the time trace. Hence, we should expect the frequency profile to be flat (or ‘white’).