# CYLINDRICAL GRAVITATIONAL WAVES: A SOURCE WITH COUNTER-ROTATING PARTICLES

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ABSTRACT

In this paper the static cylindrically symmetric exterior solutions are briefly discussed and a generalization of the concept of disposable gravitational mass per unit lenght is given. A source with counter-rotating particles is constructed to study the changes in its characteristic parameters after a gravitational pulse is emitted and the system has returned to a new static regime.
It is found that the system is physically underdetermined and that the disposable gravitational mass always decreases.

## 1. Introduction.

A great deal of work has been done on gravitational waves. In most cases the gravitational field has been studied without a direct relation to its source, even in the linearized theory. The simplest field due to a finite source is spherically symmetrical but we know, by Birkhoff's theorem, that a spherically symmetrical empty-space field is necessarily static (the Schwarzschild solution). Thus, any description of radiation from a finite system must necessarily involve, at least, three coordinates $(r,z,t)$. However we do not know, at present, of any exact solution describing gravitational waves from bounded sources. A mathematically simple situation is to consider cylindrical gravitational waves of the Einstein-Rosen[1)] type to investigate the relation between the field and its source in a system of infinite length. In this paper we consider this physically unrealistic situation constructing however a source within the kinetic theory of surface layers in GR. This source will be composed of counterrotating particles conforming an indefinite hollow cylinder that gives place to a static metric in a stationary regime.

## 2. The Coordinate System. The Static Solution.

The interior (exterior) region of the hollow cylinder will be de-

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noted by $M^-$ ($M^+$). The space-time regions $M^-$ and $M^+$ will be described by using hyperbolic canonical coordinates[2)] with the line element in the form

$$ds^2 = e^{2(\gamma-\psi)}(dr^2 - dt^2) + e^{2\psi}dz^2 + e^{-2\psi}r^2d\phi^2 \quad . \tag{1}$$

These coordinates are uniquely determined up to the (non-trivial) change of scale.

$$\begin{aligned} &r \to e^{\delta}r \;;\; t \to e^{\delta}t \;;\; z \to e^{-\delta}z \;;\; \phi \to \phi \\ &\psi \to \psi + \delta \;;\; \gamma \to \gamma \quad . \end{aligned} \tag{2}$$

The Einstein field equations in empty-space are

$$\psi_{rr} + \psi_r/r - \psi_{tt} = 0 \tag{3}$$

$$\gamma_t = 2r\,\psi_r\,\psi_t \quad ; \quad \gamma_r = r\,(\psi_r^2 + \psi_t^2) \quad . \tag{4}$$

The general static (time-independent) solution is

$$\psi = \alpha\,\ell n\, r + \beta \qquad\qquad \gamma = \alpha^2\,\ell n\, r + \epsilon \quad .$$

where $\alpha$, $\beta$ and $\epsilon$ are constants. It seems that the static solutions depend on three arbitrary constants; however the coordinate freedom represented by the transformations (2) allows us to reduce the constants to only two which may be chosen as $\alpha$ and the combination $\lambda \equiv \epsilon + \alpha^2\beta/(1-\alpha)$. It is easy to show that under the change of scale (2) the constants $\alpha$ and $\lambda$ have the important property of being invariant quantities. The invariance of the constant $\lambda$ and the transformation property of $\gamma$ suggest modifying Stachel's[3)] definition of disposable gravitational mass $M_1$ (the amount available for radiation per unit length), in the general non-static solution, by defining

$$M_1 = \lim_{r\to\infty}[\gamma - \alpha^2\,\ell n\, r + \alpha^2\beta/(1-\alpha)] \quad .$$

## 3. A Source With Counter-rotating Particles.

Within the framework of the surface layer formalism[4)] and kinetic theory i GR[5)] let us consider the space-time, $M$, as $M = M^- \cup \Sigma \cup M^+$, i.e., the junction $M^+$ to $M^-$ through the singular hypersurface $\Sigma$ (the history of the hollow cylinder). The energy-momentum tensor associated with the counter-rotating particles conforming $\Sigma$ is

$$S_{ij} = u_iu_j + p\,h_{ij} \quad ; \quad h_{ij} = \tilde{g}_{ij} + u_iu_j - v_iv_j \quad ,$$

here $\tilde{g}$ is the unique induced Lorentzian metric in $\Sigma$ given by

$$d\tilde{s}^2 = e^{2\psi_0}\,dz^2 + R^2\,e^{-2\psi_0}\,d\phi^2 - d\tau^2 \quad .$$

The indices $(i,j)$ indicate the variables $(z,\phi,\tau)$ and $v = e^{-\psi_0}\partial/\partial z$ ; $u = \partial/\partial\tau$ ; $\psi_0 = \psi|_\Sigma$ ; R is the radius of the cylinder. The explicit relations of $\eta$ and p with the energy (E) and angular momentum (J) of a single particle of proper mass $\mu$ on a circular orbit in the cylinder are

$$\eta = KE/JR \quad ; \qquad p = \eta(E^2-\mu^2)/E^2 \quad ; \qquad E^2 - (e^{\psi_0}J/R)^2 = \mu^2$$

where K is a normalization constant. The solutions in $M^+$ and $M^-$ are

$$\psi_+ = [2p/(p-\eta)]\,\ell n(r/R) + \ell n[2\pi R(\eta+p)^2/p] + \gamma_0 \quad ; \quad r \geqslant R \tag{5}$$

$$\gamma_+ = [2p/(p-\eta)]^2\ell n(r/R) + \ell n[(\eta+p)/(\eta-p)]^2 + \gamma_0 \quad ; \quad r \leqslant R \tag{6}$$

$$\psi_- = \ell n[2\pi R(\eta+p)^2/p] + \gamma_0 \equiv \psi_0 \quad ; \quad 0 \leqslant r \leqslant R \tag{7}$$

$$\gamma_- = \gamma_0 \quad ; \quad 0 \leqslant r \leqslant R \tag{8}$$

the gravitational mass per-unit of proper length of the cylinder is

$$M_{G_1} = \frac{1}{2}\alpha(\alpha-1)\ e^{-\psi_0} \quad ; \qquad \alpha \equiv 2p/(p-\eta)\ .$$

## 4. Einstein-Rosen Gravitational Radiation Pulse.

A solution of eq. (3) of the form

$$\chi(r,t) = \frac{1}{2\pi}\int_0^{t-r}\frac{f(x)\,dx}{[(t-x)^2-r^2]^{1/2}} \quad ; \quad t \geqslant r$$

where $f(x) = 0$ for $x < 0$, may be associated with a pulse of gravitational radiation.[6] By choosing $f(x)$ we can adjust intensity and duration of the pulse. Let $w(r,t)$ be the function associated with $\chi$ through eqs. (4) and $(\mathring{\psi},\mathring{\gamma})$ represents a static solution. Then a new solution of (3) and (4) are

$$\psi(r,t) = \mathring{\psi}(r) + \chi(r,t); \qquad \gamma(r,t) = \mathring{\gamma}(r) + w(r,t) + 2\alpha\chi(r,t).$$

Marder[7] has shown that if $f(x) = 0$ for $x \geqslant T \geqslant 0$, then for every r the solution becomes asymptotically (as $t \to \infty$) static again in the form $\psi \to \mathring{\psi}$ ; $\gamma \to \mathring{\gamma} - k^2$ , i-e, $\gamma$ has necessarily changed by a negative ammount and a permanent change in the source has occurred. It is precisely this change that we wish to investigate by considering the static solution with the hollow cylinder of the last section. The static exterior solution after the pulse will be

$$\bar{\psi}_+ \equiv \alpha\ \ell n\, r + \beta = \psi \quad ; \qquad \bar{\gamma}_+ \equiv \alpha^2\ \ell n\ r + \epsilon - k^2 = \gamma_+ - k^2\ .$$

There will be a corresponding change in the source, i.e.

$$\bar{\alpha} = \alpha = 2\bar{p}/(\bar{p} - \bar{\eta}) \quad ; \quad \bar{\epsilon} = \epsilon - k^2 \quad ; \quad \bar{\beta} = \beta$$ , which gives us

$$E = \bar{E} \quad ; \quad e^{\psi_0} = e^{\bar{\psi}_0} (R/\bar{R})^{\alpha} \quad ; \quad (\bar{J}/J) = (\bar{R}/R)^{1-\alpha} \quad ;$$

$$(\bar{M}_{G_1}/M_{G_1}) = (R/\bar{R})^{\alpha} \quad . \tag{9}$$

To complete our study of the change in the source parameters we need an additional hypothesis to establish, for instance, a relation between R and $\bar{R}$. Assuming, e.g., that the normalization constant K remains unchanged, i.e., $K = \bar{K}$, from (9) and the definition of K we obtain

$$(\alpha^2 - 1)\ \ell n(\bar{R}/R) = k^2$$

therefore if $|\alpha| < 1$ (the weak field limit is obtained when $p \ll \eta$, and thus $|\alpha|$ may be considered small), we have

$$\bar{R} < R \quad ; \quad \bar{J} < J \quad ; \quad \bar{M}_{G_1} < M_{G_1} \quad . \tag{10}$$

The physical suitability of the hypothesis $K = \bar{K}$ may be justified recalling first that the system under consideration is infinite and consequently has not a well defined number of particles and secondly, the invariance of a normalization constant like K is a familiar hypothesis in similar cases. However, it is important to realize that this hypothesis is crucial to obtain the result (10). For instance, if we were to assume that the total number of particles per unit of proper length is conserved, we would have obtained $(\bar{\eta}/\eta) = (R/\bar{R})\ e^{\bar{\psi}_0 - \psi_0}$, and therefore $\bar{\gamma}_0 = \gamma_0$, which implies that $\bar{R} > R$; $\bar{J} > J$ and $\bar{M}_{G_1} > M_{G_1}$. However, in all these cases the disposable gravitational mass in the static regime is $M_1 = \lambda$ and it changes according to

$$\bar{M}_1 = M_1 - k^2 \quad ,$$

showing that $M_1$, at least in these cases, is a good measure of the amount of mass available for radiation.

## 5. References


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