Finite Element Methods for a Model for Full Waveform Acoustic Logging

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Dedicated to Professor Leslie Fox on the occasion of his seventieth birthday

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A model is defined to simulate the propagation of waves in a radially symmetric, isotropic, composite system consisting of a fluid-filled well bore $Q_f$ through a fluid-saturated porous solid $Q_p$. Biot's equations of motion are chosen to describe the propagation of waves in $Q_p$, while the standard equation of motion for compressible inviscid fluids is used for $Q_f$, with appropriate boundary conditions at the contact surface between $Q_f$ and $Q_p$. Also, absorbing boundary conditions for the artificial boundaries of $Q_p$ are derived for the model, their effect being to make them transparent for waves arriving normally.

First, results on the existence and uniqueness of the solution of the differential problem are given and then a discrete-time, explicit finite element procedure is defined and analysed, with finite element spaces suited for radially symmetric problems being used for the spatial discretisation.

1. Introduction

We consider the problem of acoustic and elastic wave propagation in a cylindrical fluid-filled borehole $\Omega_f$ through a fluid-saturated porous solid $\Omega_p$. The problem arises naturally in acoustic well-logging. A compressional point source is excited at a point on the centreline of the borehole, and the energy transmitted through the fluid in the borehole and through the surrounding formations is recorded by receivers located in the well bore, both above and below the source, to obtain what is known as a full waveform acoustic log [19]. Here, in order to simplify the problem we have assumed that the whole system $\Omega = \Omega_f \cup \Omega_p$ is isotropic and radially symmetric around the $z$-axis, located at the centre of the borehole.

To describe the propagation of waves in $\Omega_p$ we have chosen Biot's equations of motion, while in $\Omega_f$ we have used the standard equation of motion for compressible, inhomogeneous, inviscid fluids. Appropriate absorbing boundary

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conditions for the artificial boundaries of $\Omega_p$ are derived, making them transparent to waves arriving normally. The same type of absorbing boundary conditions are used for the artificial boundaries of $\Omega_f$. For the surface contact between $\Omega_f$ and $\Omega_p$ we have chosen the boundary condition suggested in [14], which represents a way of including the effects of the mud cake in the wave field. The special cases of an open or closed interface are also considered in the model.

This paper is related to several previous works on the subject. The theory of propagation of waves in fluid-saturated porous media was formulated by Biot in several classic papers [1, 2]. Results on the existence, uniqueness and finite element approximation of the solution of Biot’s equations were given in [15, 17]. Also, mixed finite elements for radially symmetric three-dimensional problems were presented and analysed in [12]. Finite element methods for wave propagation in systems composed of elastic solids with imbedded fluid-saturated porous media were given and analysed in [11, 16], and some numerical results using those algorithms were presented in [4].

Synthetic full waveform acoustic logs have already been obtained using different techniques. In [14] the problem was treated assuming that the system $\Omega$ is homogeneous in depth and the solution was obtained via Fourier-transform techniques and numerical integration. The same approach was used in [3] but with $\Omega_p$ being an elastic solid. In [18], the system $\Omega$ was allowed to be inhomogeneous in depth with $\Omega_p$ being again an elastic solid; an approximate solution was computed using finite-difference techniques.

The organisation of the paper is as follows. In §2 we present the model by giving the partial differential equations and the initial and boundary conditions. In §3 we derive the weak form of the model and then present the results on the existence and uniqueness of the solution of the differential problem. In §4 we describe the finite element spaces used for the spatial discretisation and then formulate an explicit finite element procedure by using a mass-lumping quadrature rule for the first- and second-order time-derivative terms in the weak formulation. Results on the stability and convergence of the method are also given. Finally in §5 we derive the absorbing boundary conditions used for the artificial boundaries of $\Omega_p$.

2. The model

We shall consider the propagation of waves in a fluid-filled borehole $\Omega_f$ surrounded by a fluid-saturated porous medium $\Omega_p$. For simplicity the whole system $\Omega = \Omega_f \cup \Omega_p$ will be assumed to be isotropic and radially symmetric around the $z$-axis, located at the centre of the borehole. The system is naturally described using cylindrical coordinates $(r, \theta, z)$. Without loss of generality, the artificial top and bottom boundaries of $\Omega_f$ and $\Omega_p$ can be chosen to be the sets

$$I_1 = \{ (r, \theta, z) \in \partial \Omega_f : 0 \leq r \leq R_f^T, 0 \leq \theta < 2\pi, z = 0 \},$$

and

$$I_2 = \{ (r, \theta, z) \in \partial \Omega_p : R_p^T \leq r \leq R_p, 0 \leq \theta < 2\pi, z = 0 \}$$

and

$$I_{21} = \{ (r, \theta, z) \in \partial \Omega_p : R_f^T \leq r \leq R_p, 0 \leq \theta < 2\pi, z = z_b \}$$

and

$$I_{22} = \{ (r, \theta, z) \in \partial \Omega_p : R_f^T \leq r \leq R_p, 0 \leq \theta < 2\pi, z = z_b \}.$$
Also, the artificial exterior boundary of $\Omega_p$ (and $\Omega$) can be taken to be

$$\Gamma_2 = \{(r, \theta, z) \in \partial \Omega_p : r = R_p, 0 \leq \theta < 2\pi, 0 \leq z \leq z_B\}.$$  

Let $\Gamma_3$ denote the surface contact between $\Omega_f$ and $\Omega_p$, which may have arbitrary shape in order to allow variations in the diameter of the borehole along the $z$-direction. A vertical cross-section of $\Omega$ for any fixed $\theta = \theta_0$ is shown in Fig. 1.

Assume cylindrical symmetry, and let $u_1 = (u_{1r}, 0, u_{1z})$ be the fluid displacement in $\Omega_f$, let $u_2 = (u_{2r}, 0, u_{2z})$ be the solid displacement in $\Omega_p$, and let $u_3 = (\bar{u}_{3r}, 0, \bar{u}_{3z})$ be the (averaged) fluid displacement in $\Omega_p$. Let

$$u_3 = \phi(x)(\bar{u}_3 - u_2) = (u_{3r}, 0, u_{3z}),$$

where $\phi(x)$ is the effective porosity, and set $u = (u_1, u_2, u_3)$. Here $u_{ki}$ represents the displacement in the $i$-direction for $k = 1, 2, 3$. Next, because of the cylindrical symmetry assumption, the physical components of the strain tensor $\varepsilon(u_2)$ in the solid part of $\Omega_p$ are given [8, p. 114] by

$$\varepsilon_r(u_2) = \frac{\partial u_{2r}}{\partial r}, \quad \varepsilon_{\theta\theta}(u_2) = \frac{u_{2\theta}}{r}, \quad \varepsilon_{zz}(u_2) = \frac{\partial u_{2z}}{\partial z},$$

$$\varepsilon_{rr}(u_2) = \frac{1}{r} \left( \frac{\partial u_{2r}}{\partial \theta} + \frac{\partial u_{2\theta}}{\partial r} \right),$$

$$\varepsilon_{\theta\theta}(u_2) = \frac{1}{r} \left( \frac{\partial u_{2r}}{\partial \theta} + \frac{\partial u_{2\theta}}{\partial r} - \frac{u_{2r}}{r} \right) = 0,$$

$$\varepsilon_{\theta z}(u_2) = \frac{1}{r} \left( \frac{\partial u_{2\theta}}{\partial z} + \frac{1}{r} \frac{\partial u_{2z}}{\partial \theta} \right) = 0.$$
Also, note that
\[ \nabla \cdot u_2 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{1}{r} \frac{\partial (ru_2)}{\partial r} + \frac{\partial u_{2z}}{\partial z}. \]

Let \( \tau(u_2, u_3) \) and \( p(u_2, u_3) \) denote the total stress tensor and the fluid pressure in \( \Omega_p \), respectively. Then the stress-strain relations in \( \Omega_p \) can be written as follows [2]:
\[
\begin{align*}
\tau_{rr}(u_2, u_3) &= A \nabla \cdot u_2 + 2N \varepsilon_{rr}(u_2) + Q \nabla \cdot u_3, \\
\tau_{\theta\theta}(u_2, u_3) &= A \nabla \cdot u_2 + 2N \varepsilon_{\theta\theta}(u_2) + Q \nabla \cdot u_3, \\
\tau_{zz}(u_2, u_3) &= A \nabla \cdot u_2 + 2N \varepsilon_{zz}(u_2) + Q \nabla \cdot u_3, \\
\tau_{rz}(u_2) &= 2N \varepsilon_{rz}(u_2), \\
\tau_{\theta z} &= 0, \\
p(u_2, u_3) &= -Q \nabla \cdot u_2 - H \nabla \cdot u_3.
\end{align*}
\]

In these expressions, the coefficients \( A, N, Q \) and \( H \) are assumed to be functions of \( r \) and \( z \) alone.

Next, the strain-energy density \( W_p(u_2, u_3) \) in \( \Omega_p \) is given [2] by
\[ W_p(u_2, u_3) = \frac{1}{2} \left[ \tau_{rr} \varepsilon_{rr} + \tau_{\theta\theta} \varepsilon_{\theta\theta} + \tau_{zz} \varepsilon_{zz} + 2 \tau_{rz} \varepsilon_{rz} - p \nabla \cdot u_3 \right]. \] (2.2)

Since \( W_p \) has to be a quadratic, positive-definite form in \( \varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \varepsilon_{rz} \) and \( \nabla \cdot u_3 \), it is easily seen that the coefficients \( A, N, Q \) and \( H \) must satisfy the conditions
\[
\begin{align*}
N(r, z) > 0, & \quad H(r, z) > 0, \quad (A + \frac{3}{2}N)(r, z) > 0, \\
(A + \frac{3}{2}N - Q^2/H)(r, z) > 0, & \quad (r, \theta, z) \in \overline{\Omega_p} = \Omega_p \cup \partial \Omega_p.
\end{align*}
\] (2.3)

Also, \( dW_p \) must be an exact differential, so that
\[
\begin{align*}
\tau_{rr} &= \frac{\partial W_p}{\partial \varepsilon_{rr}}, & \tau_{\theta\theta} &= \frac{\partial W_p}{\partial \varepsilon_{\theta\theta}}, & \tau_{zz} &= \frac{\partial W_p}{\partial \varepsilon_{zz}}, \\
\tau_{rz} &= \frac{\partial W_p}{\partial \varepsilon_{rz}}, & p &= \frac{\partial W_p}{\partial (-\nabla \cdot u_3)}.
\end{align*}
\] (2.4)

Let \( E_p(r, z) \in \mathbb{R}^{5 \times 5} \) be the symmetric, positive-definite matrix associated with \( W_p \) and set
\[ Y(u_2, u_3) = (\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \nabla \cdot u_3, \varepsilon_{rz})^T, \]
so that
\[ W_p(r, z) = \frac{1}{2} [E_p Y, Y], \]
where \( [,]_e \) denotes the usual scalar product in \( \mathbb{R}^n \).

For any matrix \( D(r, z) \in \mathbb{R}^{n \times n} \), let \( \lambda_{\min}(D(r, z)) \) and \( \lambda_{\max}(D(r, z)) \) denote the minimum and maximum eigenvalues of \( D(r, z) \) and set
\[ \lambda_{\min}(D) = \inf_{r,z} \lambda_{\min}(D(r, z)), \quad \lambda_{\max}(D) = \sup_{r,z} \lambda_{\max}(D(r, z)). \]
Under the conditions (2.3),

\[ 0 < \lambda_{\text{min}}(E_p) \leq \lambda_{\text{max}}(E_p) < \infty, \]

and consequently,

\[
W_p(r, z) \geq \frac{\lambda_{\text{min}}(E_p)}{2} ((\varepsilon_{rr})^2 + (\varepsilon_{\theta\theta})^2 + (\varepsilon_{zz})^2 + (\nabla \cdot u_3)^2)
\]

\[
\geq \frac{\lambda_{\text{min}}(E_p)}{4} ((\varepsilon_{rr})^2 + (\varepsilon_{\theta\theta})^2 + (\varepsilon_{zz})^2 + 2(\varepsilon_{rz})^2 + (\nabla \cdot u_3)^2). \tag{2.5}
\]

Next, let \( \rho = \rho(r, z) \) denote the total mass density of bulk material in \( \Omega_p \) and let \( \rho_f = \rho_f(r, z) \) be the mass density of fluid both in \( \Omega_f \) and \( \Omega_p \). Also, let \( g = g(r, z) \) be a mass-coupling parameter between fluid and solid in \( \Omega_p \) [2]. Assume that

\[
\rho_g(r, z) - \rho_f^2(r, z) > 0, \quad (r, \theta, z) \in \Omega_p,
\]

which is a necessary and sufficient condition for the kinetic-energy density in \( \Omega_p \) to be positive.

Let \( \mu = \mu(r, z) \) denote the fluid viscosity and let \( k = k(r, z) \) denote the (scalar) rock permeability in \( \Omega_p \). Both \( \mu \) and \( k \) will be assumed to be bounded above and below by positive constants.

Finally, let \( A_f = A_f(r, z) \) denote the incompressibility modulus of the fluid in \( \Omega_f \), assumed to be bounded above and below by positive constants:

\[
0 < A_f < A_f^* < \infty.
\]

Then, we consider the following problem. Let

\[
u^0_i(r, z) = (u^0_{1r}, 0, u^0_{1z}), \quad v^0_i = (v^0_{1r}, 0, v^0_{1z}), \quad f_i = (f_{1r}, 0, f_{1z})
\]

be given for \( (r, \theta, z) \in \Omega_f \) and let

\[
u^0_2(r, z) = (u^0_{2r}, 0, u^0_{2z}), \quad u^0_i(r, z) = (u^0_{3r}, 0, u^0_{3z}),
\]

\[
u^0_2(r, z) = (v^0_{2r}, 0, v^0_{2z}), \quad v^0_i(r, z) = (v^0_{3r}, 0, v^0_{3z}),
\]

and

\[
f_i = (f_{2r}, 0, f_{2z}), \quad f_i = (f_{3r}, 0, f_{3z})
\]

be given for \( (r, \theta, z) \in \Omega_p \). Then we want to find \( u(r, z, t) = (u_1, u_2, u_3), \) \( t \in J = (0, T) \), such that

\[
\begin{align*}
(i) \quad \rho_f \frac{\partial^2 u_{1r}}{\partial t^2} - \frac{\partial}{\partial r} (A_f \nabla \cdot u_i) &= f_{1r}(r, z, t), \\
(ii) \quad \rho_f \frac{\partial^2 u_{1z}}{\partial t^2} - \frac{\partial}{\partial z} (A_f \nabla \cdot u_i) &= f_{1z}(r, z, t)
\end{align*}
\]

\[
\tag{2.7a}
\]
for \((r, \theta, z, t) \in \Omega_f \times J,\) and

\[
\begin{align*}
(iii) \quad & \frac{\partial^2 u_{2r}}{\partial t^2} + \rho_f \frac{\partial^2 u_{3r}}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{rr}(u_2, u_3) \right) \\
& \quad - \frac{\partial \tau_{zr}(u_2)}{\partial z} + \frac{\tau_{88}(u_2, u_3)}{r} = f_{2r}(r, z, t), \\
(iv) \quad & \frac{\partial^2 u_{23}}{\partial t^2} + \rho_f \frac{\partial^2 u_{33}}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \tau_{zz}(u_2) \right) - \frac{\partial}{\partial z} \tau_{z2}(u_2, u_3) = f_{23}(r, z, t), \\
(v) \quad & \rho_f \frac{\partial^2 u_{2r}}{\partial t^2} + g \frac{\partial^2 u_{3r}}{\partial t^2} + \frac{\mu}{k} \frac{\partial u_{3r}}{\partial t} + \frac{\partial}{\partial r} p(u_2, u_3) = f_3(r, z, t), \\
(vi) \quad & \rho_f \frac{\partial^2 u_{23}}{\partial t^2} + g \frac{\partial^2 u_{33}}{\partial t^2} + \frac{\mu}{k} \frac{\partial u_{33}}{\partial t} + \frac{\partial}{\partial z} p(u_2, u_3) = f_{32}(r, z, t)
\end{align*}
\]

(2.7b)

for \((r, \theta, z, t) \in \Omega_p \times J,\) with boundary conditions

\[
\begin{align*}
(i) \quad & -A_f \nabla \cdot u_1 = (\rho_p A_f)^{1/2} \frac{\partial u_1}{\partial t} \cdot v_f, \quad (r, \theta, z, t) \in \Gamma_1 \times J, \\
(ii) \quad & (-\tau_v \cdot v, -\tau_v \cdot \chi^1, -\tau_v \cdot \chi^2, p)^T \\
& \quad = B \left( \frac{\partial u_2}{\partial t} \cdot v, \frac{\partial u_2}{\partial t} \cdot \chi^1, \frac{\partial u_2}{\partial t} \cdot \chi^2, \frac{\partial u_3}{\partial t} \cdot v \right)^T, \\
& \quad (r, \theta, z, t) \in (\Gamma_2 \cup \Gamma_2) \times J = \Gamma_2 \times J, \\
(iii) \quad & \tau_v + A_f \nabla \cdot u_1 v_f = 0, \quad (r, \theta, z, t) \in \Gamma_3 \times J, \\
(iv) \quad & (u_2 + u_3) \cdot v + u_1 \cdot v_f = 0, \quad (r, \theta, z, t) \in \Gamma_3 \times J, \\
(v) \quad & -p + m \frac{\partial u_3}{\partial t} \cdot v = A_f \nabla \cdot u_1, \quad (r, \theta, z, t) \in \Gamma_3 \times J,
\end{align*}
\]

(2.8)

and initial conditions

\[
\begin{align*}
(i) \quad & u_1(r, z, 0) = u_1^0(r, z), \quad (r, \theta, z) \in \Omega_f, \\
(ii) \quad & (u_2, u_3)(r, z, 0) = (u_2^0, u_3^0)(r, z), \quad (r, \theta, z) \in \Omega_p, \\
(iii) \quad & \frac{\partial u_1}{\partial t}(r, z, 0) = v_1^0(r, z), \quad (r, \theta, z) \in \Omega_f, \\
(iv) \quad & \frac{\partial (u_2, u_3)}{\partial t}(r, z, 0) = (v_2^0, v_3^0)(r, z), \quad (r, \theta, z) \in \Omega_p.
\end{align*}
\]

(2.9)

In the above, \(v_i = (v_{ir}, v_i\theta, v_iz) = (v_{ir}, 0, v_{iz}), \quad i = f, p,\) denotes the unit outward normal along \(\partial \Omega_i,\) and \(\chi^m_p, m = 1, 2,\) denotes orthogonal unit tangent vectors along \(\partial \Omega_p.\) Also, \(\tau_v\) denotes the stress tensor on \(\partial \Omega_p\) and \(\tau_v \cdot v_p \) and \(\tau v_p \cdot \chi^m, m = 1, 2,\) are the normal and two tangent components of \(\tau v_p\) on \(\partial \Omega_p.\)

Equations (2.7a) are the standard equations of motion for compressible, inviscid, inhomogeneous fluids, while equations (2.7b) are Biot's equations of motion for the fluid-saturated porous medium \(\Omega_p, [1, 2]\). The boundary condition (2.8.i) is simply the equation of momentum for \(\Gamma_1,\) so that waves arriving
normally to $I_2$ will be absorbed completely (that is, passed through transparently). Equation (2.8.ii) is an absorbing boundary condition for the artificial boundary $I_2$ of $Q_p$; this relation is derived in §5. Again, its effect is to absorb the energy of waves arriving normally to $I_2$. The matrix $B(r, z) \in \mathbb{R}^{4 \times 4}$ in the right-hand side of (2.8.ii) is symmetric and positive definite. Equation (2.8.iii) states the continuity of the normal stress and the vanishing of tangential stresses along $I_3$, while (2.8.iv) expresses the continuity of the normal displacement on $I_3$.

Finally, (2.8.v) relates the fluid pressure on both sides of $I_3$. This boundary condition is suggested in [14] to describe the behaviour of the mud cake using the non-negative coefficient $m = m(z)$ representing a surface impedance. The analysis of the model will be carried out for the case in which $0 < m_\ast \leq m(z) \leq m^\ast < \infty$, and we shall indicate briefly the change in the argument for the limit cases $m = 0$ and $m = +\infty$ corresponding to an open or sealed interface, respectively. Note that in the case of an open interface ($m = 0$), (2.8.v) simply states the continuity of the fluid pressure on $I_3$. Such a boundary condition was analysed in [10] and was shown to be energy-flux preserving. For a sealed interface ($m = +\infty$), it is necessary that

$$u_3 \cdot v_p = 0, \quad (r, \theta, z, t) \in I_3 \times J. \tag{2.10}$$

In this case (2.8.v) should be replaced by (2.10), and (2.8.iv) reduces to

$$u_2 \cdot v_p + u_1 \cdot v_f = 0, \quad (r, \theta, z, t) \in I_3 \times J.$$

3. The existence and uniqueness results

For $\Omega_f = \Omega_f$ or $\Omega_p$, let

$$(\varphi, \psi)_\Omega = \int_{\Omega} \varphi(r, \theta, z) \psi(r, \theta, z) r \, dr \, d\theta \, dz$$

and

$$\|\varphi\|_{0, \Omega} = [(\varphi, \varphi)_\Omega]^\frac{1}{2}$$

denote the inner product and norm in $L^2(\Omega_f)$. For any $\Gamma \subset \partial \Omega_f$ let

$$\langle v, w \rangle_\Gamma = \int_{\Gamma} vw \, d\sigma$$

denote the inner product in $L^2(\Gamma)$, where $d\sigma$ is the surface measure on $\Gamma$. Also, if $\varphi = (\varphi_r, \varphi_\theta, \varphi_z)$ and $\psi = (\psi_r, \psi_\theta, \psi_z)$, we shall denote by

$$(\varphi, \psi)_\Gamma = (\varphi_r, \psi_r)_\Gamma + (\varphi_\theta, \psi_\theta)_\Gamma + (\varphi_z, \psi_z)_\Gamma$$

and

$$\|\varphi\|_{0, \Omega} = [(\varphi, \varphi)_\Gamma]^\frac{1}{2}$$

denote the inner product and norm in $L^2(\Omega_f)^3$.

Next, let

$$H(\text{div}, \Omega_f) = \{ \varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in L^2(\Omega_f)^3: \nabla \cdot \varphi \in L^2(\Omega_f) \}.$$
provided with the natural norm
\[ \| \varphi \|_{H(\text{div}, \Omega)} = [\| \varphi \|_{0, \Omega}^2 + \| \nabla \cdot \varphi \|_{0, \Omega}^2]^{\frac{1}{2}}. \]

Set
\[ \tilde{H}(\text{div}, \Omega) = \{ \varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in H(\text{div}, \Omega): \varphi_\theta = 0 \}, \]
which is a closed subspace of $H(\text{div}, \Omega)$. Note that $\varphi \in H(\text{div}, \Omega_r)$ implies that $\varphi_r|_{r=0} = 0$. Also, set
\[ \tilde{H}^1(\Omega_p)^3 = \{ \varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in H^1(\Omega_p)^3: \varphi_\theta = 0, \partial \varphi_r / \partial \theta = \partial \varphi_z / \partial \theta = 0 \} \]
\[ = \{ \varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in H^1(\Omega_p)^3: \varphi_\theta = 0, \varepsilon_\theta \varphi = \varepsilon_\theta \varphi_\theta (\varphi_\theta = 0) \}. \]

Note that, for any $\varphi \in \tilde{H}^1(\Omega_p)^3$, standard calculus shows that in $\mathbb{R}^3$ with cylindrical symmetry,
\[ \| \varphi \|_{1, \Omega_p} = \left[ \int_{\Omega_p} \left( (\partial \varphi_r / \partial r)^2 + (\partial \varphi_z / \partial z)^2 + \left( \frac{\partial \varphi_r}{r} \right)^2 \right) r \, dr \, d\theta \, dz \right]^{\frac{1}{2}}. \]

It is clear that $\tilde{H}^1(\Omega_p)^3$ is a closed subspace of $H^1(\Omega_p)^3$.

Next, let $\tilde{V} = \tilde{H}(\text{div}, \Omega_r) \times \tilde{H}^1(\Omega_p)^3 \times \tilde{H}(\text{div}, \Omega_p)$, which is a separable Hilbert space under the norm
\[ \| v \|_{\tilde{V}} = \left[ \| v_1 \|_{\tilde{H}(\text{div}, \Omega_r)}^2 + \| v_2 \|_{\tilde{H}^1(\Omega_p)^3}^2 + \| v_3 \|_{\tilde{H}(\text{div}, \Omega_p)}^2 \right]^{\frac{1}{2}}. \]

Since the boundary condition (2.8.iv) will be imposed strongly, we shall restrict the admissible test functions to the set
\[ V = \{ v = (v_1, v_2, v_3) \in \tilde{V}: (v_2 + v_3 - v_1) \cdot v_f = 0 \text{ on } \Gamma_3 \}; \]
$V$ is a closed, separable subspace of $\tilde{V}$ (with the same norm).

The weak form of problem (2.7), (2.8), and (2.9) is obtained as usual by testing equations (2.7) against any admissible function $v = (v_1, v_2, v_3) \in V$, using integration by parts and applying the boundary conditions (2.8.i), (2.8.ii), (2.8.iii), and (2.8.v). In doing so, we obtain
\[ \left( \rho_f \frac{\partial^2 u_1}{\partial t^2}, v_1 \right)_f + \left( \mathcal{A} \frac{\partial^2 (u_2, u_3)}{\partial t^2}, (v_2, v_3) \right)_p + \left( \mathcal{C} \frac{\partial (u_2, u_3)}{\partial t}, (v_2, v_3) \right)_p + A(u, v) \]
\[ + \left( \rho_f A_f \frac{\partial u_1}{\partial t}, v_1 \cdot v_f \right)_f + \left( B \left( \frac{\partial u_2}{\partial t}, v_p, \frac{\partial u_2}{\partial t}, \chi_p, \frac{\partial u_2}{\partial t}, \chi^2_p, \frac{\partial u_3}{\partial t}, \nu_p \right), \nu_p \right)_f \]
\[ + \left( m \frac{\partial u_3}{\partial t}, v_3 \cdot v_p \right)_f = (f_1, v_1) + ((f_2, f_3), (v_2, v_3))_p, \quad v = (v_1, v_2, v_3) \in V, t \in J. \] \hspace{1cm} (3.1)

Here $\mathcal{A}(r, z)$ and $\mathcal{C}(r, z)$ are matrices in $\mathbb{R}^{4 \times 4}$ given by
\[ \mathcal{A} = \begin{pmatrix} \rho f & \rho f I \\ \rho f I & \rho f \end{pmatrix}, \quad \mathcal{C} = \mu k^{-1} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \]
I being the identity matrix in \( \mathbb{R}^{2 \times 2} \). Note that \( \mathbf{C} \) is non-negative and \( \mathcal{A} \) is positive definite, thanks to (2.6). Also, \( \Lambda(v, w) \) is the symmetric, bilinear form defined on \( \tilde{V} \) by

\[
\Lambda(v, w) = (A_f \cdot v_1, \nabla \cdot w_1) + (\tau_{rr}(v_2, v_3), \varepsilon_{rr}(w_2))_p
\]

\[
+ (\tau_{oo}(v_2, v_3), \varepsilon_{oo}(w_2))_p + (\tau_{zz}(v_2, v_3), \varepsilon_{zz}(w_2))_p
\]

\[
+ 2(\tau_{rz}(v_2, v_3), \varepsilon_{rz}(w_2))_p
\]

\[
- (p(v_2, v_3), \nabla \cdot w_3)_p
\]

for \( v, w \in \tilde{V} \).

Note that combining (2.2), (2.5), and Korn's second inequality [6, 7, 13] implies that

\[
\Lambda(v, v) \geq A_f \| \nabla \cdot v_1 \|_{0, \Omega}^2 + \frac{\lambda_{\min}(E_f)}{2} \int_{\Omega_p} [((\varepsilon_{rr}(v_2))^2 + (\varepsilon_{oo}(v_2))^2)
\]

\[
+ (\varepsilon_{zz}(v_2))^2 + 2(\varepsilon_{rz}(v_2)^2 + (\nabla \cdot v_3)^2)] \, dr \, d\theta \, dz
\]

\[
\geq c_1 \| v_1 \|^2_{0, \Omega} + c_2 (\| v_2 \|^2_{0, \Omega} + \| (v_2, v_3) \|^2_{0, \Omega}), \quad v \in \tilde{V}. \quad (3.2)
\]

Let \( c_2 \) be any fixed constant and let \( \Lambda_\gamma \) be the bilinear symmetric form defined over \( \tilde{V} \) by

\[
\Lambda_\gamma(v, w) = \Lambda(v, w) + \gamma[(v_1, w_1)_f + ((v_2, v_3), (w_2, w_3))_p].
\]

Then \( \Lambda_\gamma \) is \( \tilde{V} \)-continuous and \( \tilde{V} \)-coercive.

Next, set

\[
Q_i^2 = \left\| \frac{\partial f_i}{\partial r} \right\|_{L^2(J, L^2(\Omega_p))^2}^2 + \left\| \frac{\partial^2 f_i}{\partial r^2} \right\|_{L^2(J, L^2(\Omega_p))^2}^2
\]

\[
G_0^0 = \| u_0^0 \|^2_{0, \Omega} + \| (u_2^0, u_3^0) \|^2_{0, \Omega} + \| v_0^0 \|^2_{\tilde{V}} + \| (f_1(0), f_2(0), f_3(0)) \|^2_{0, \Omega} + 1.
\]

The well-posedness of problem (2.7), (2.8), and (2.9) follows from the following theorem.

**Theorem 3.1** Let \( f = (f_1, f_2, f_3) \), \( u^0 = (u_0^0, u_2^0, u_3^0) \) and \( v^0 = (v_1^0, v_2^0, v_3^0) \) be given and such that \( G_0^0 < \infty \), \( Q_i^2 < \infty \), \( i = 0, 1 \). Assume that \( \Gamma_3 \) is of class \( C^m \) for some integer \( m \geq 2 \). Also, assume that

- support \((u_1^0) \cap \Omega_p \subseteq \Omega_f\), support \((v_1^0) \cap \Omega_p \subseteq \Omega_f\),
- support \((u_2^0, u_3^0) \subseteq \Omega_p\), support \((v_2^0, v_3^0) \subseteq \Omega_p\).

Then there exists a unique solution \( u(r, z, t) \) of problem (2.7), (2.8), and (2.9) such that \( u, \partial u / \partial t \in L^\infty(J, V) \); \( \partial^2 u / \partial r^2 \in L^2(J, L^2(\Omega_p))^2 \); and \( \partial^2 (u_1, u_3) / \partial r \partial z \in L^2(J, L^2(\Omega_p)). \)

**Proof.** Let

\[
\tilde{H}^2(\Omega_f)^3 = \{ \phi \in H^2(\Omega_f)^3 : \phi|_{r=0} = 0, \phi_0 = 0, \partial \phi / \partial \theta = \partial \phi / \partial \theta = 0 \},
\]

\[
\tilde{H}^2(\Omega_p)^3 = \{ \phi \in H^2(\Omega_p)^3 : \phi_0 = 0, \partial \phi / \partial \theta = \partial \phi / \partial \theta = 0 \},
\]

and set

\[
E = \tilde{H}^2(\Omega_f)^3 \times \tilde{H}^2(\Omega_p)^3 \times \tilde{H}^2(\Omega_p)^3.
\]

Clearly, \( E \subseteq \tilde{V} \) and the argument given in [16] can be used here to show that \( E \cap V \) is dense in \( V \). The compactness argument given in [15, 16] can be used with minor modifications to obtain the conclusions of the theorem.
In the case in which the contact surface \( F_3 \) between \( \Omega_f \) and \( \Omega_p \) is known just to be Lipchitz continuous, the following existence and uniqueness theorem holds, its proof being similar to that of Theorem 3.1.

**Theorem 3.2** Let \( f = (f_1, f_2, f_3) \) be given and such that \( Q_i < \infty, \ i = 0, 1 \). Assume that \( u^0 = v^0 = 0 \) and that \( F_3 \) is Lipchitz continuous. Then there exists a unique solution \( u(r, z, t) \) of problem (2.7), (2.8), and (2.9) such that \( u, \partial u/\partial t \in L^\infty(J, V) \); \( \partial^2 u_1/\partial r^2 \in L^2(J, L^2(\Omega_f)^2) \); and \( \partial^2 (u_2, u_3)/\partial r^2 \in L^\infty(J, L^2(\Omega_p)^4) \).

Finally, let us indicate the modifications needed to treat the cases of an open or a sealed interface \( F_3 \). For the open interface \( (m = 0) \) the original space \( V \) is adequate. For the sealed interface \( (m = +\infty) \) the space \( V \) should be chosen to be

\[
V = \{ v = (v_1, v_2, v_3) \in \tilde{V} : (v_2 - v_1) \cdot v_f = 0, v_3 \cdot v_f = 0 \text{ on } F_3 \}.
\]

Thus, in both cases the weak form (3.1) remains formally unchanged, except that the last term in the left-hand side disappears. Also, the conclusions of Theorems 3.1 and 3.2 remain valid.

### 4. An Explicit Finite Element Procedure

For \( 0 < h < 1 \), let \( \tau_h = \tau_h(\Omega_f) \) and \( \tau_h = \tau_h(\Omega_p) \) be quasiregular partitions of \( \Omega_f \) and \( \Omega_p \) into elements generated by the rotation around the \( z \)-axis of rectangles in the \( (r, z) \)-variables of diameter bounded by \( h \). Set \( \tau_h = \tau_h \cup \tau_h^p \). Since the boundary condition (2.8.iv) will be imposed strongly on the finite element spaces to be used for the spatial discretisation, the partitions \( \tau_h \) and \( \tau_h^p \) will be assumed to be compatible along the contact surface \( F_3 \) in the following sense. For any vertical cross-section \( r_h \cap \{ \theta = \theta_0 \} \) of \( \tau_h \), if \( R_f \) is a rectangle in \( r_h \cap \{ \theta = \theta_0 \} \) such that one edge \( e \) of \( R_f \) is contained in \( F_3 \), then \( e \) is also an edge of some rectangle \( R_p \) in \( r_h \cap \{ \theta = \theta_0 \} \). Let \( P_{1,1}(r, z) \) denote the bilinear polynomials in the \( (r, z) \)-variables and set

\[
M_h = \{ \varphi = (\varphi_r, 0, \varphi_z) \in C^0(\Omega_p) : \varphi_r \in rP_{1,1}(r, z) \text{ and } \varphi_z \in P_{1,1}(r, z) \}.
\]

Then, \( M_h \subset H^1(\Omega_p)^3 \).

The \( r \)-component of \( \varphi \) is multiplied by \( r \) in order to ensure that all components of the strain tensor of \( \varphi \) remain polynomials in \( r \) and \( z \). It does not affect the approximation property

\[
\inf \{ ||v - \varphi||_{0, \Omega_p} + h ||v - \varphi||_{1, \Omega_p} \leq ch^s ||v||_{s, \Omega_p}, \quad s = 1, 2; \quad (4.1)
\]

this result is proved in [12].

Let \( W_h(\Omega_i), i = f, p \), be the vector part of the lowest-order mixed finite element space associated with \( \tau_h \) defined by Morley [12]. Away from \( r = 0 \), the elements in \( W_h(\Omega_i) \) are locally of the form \( (ar^{-1} + br, 0, c + dz) \), while the innermost elements near \( r = 0 \) have the local form \( (br, 0, c + dz) \). Globally the elements must lie in \( H(\text{div}, \Omega_i) \), \( i = f \) or \( p \), as appropriate. Note that the divergence of each
element is piecewise constant. It is shown in [12] that
\[
\begin{align*}
(i) & \quad \inf_{\varphi \in W_h(\Omega)} \| u - \varphi \|_{H(\text{div}, \Omega)} \leq c h \left( \| u \|_{1, \Omega} + \| \nabla \cdot u \|_{1, \Omega} \right), \\
(ii) & \quad \inf_{\varphi \in W_h(\Omega)} \| u - \varphi \|_{0, \Omega} \leq c h \| u \|_{1, \Omega}.
\end{align*}
\]

(4.2)

Let \( \tilde{V}_h = W_h(\Omega_f) \times \mathcal{M}_h \times W_h(\Omega_p) \) and set \( V_h = \{ v \in \tilde{V}_h : (v_2 + v_3 - v_1) \cdot v_f = 0 \text{ on } \Gamma_f \} \). Then \( V_h \subset V \) and it follows from (4.1) and (4.2) that
\[
\inf_{\varphi \in V_h} \| u_1 - \varphi_1 \|_{0, \Omega_f} + \| (v_2, v_3) - (\varphi_2, \varphi_3) \|_{0, \Omega_f} \leq c h \left[ \| u_1 \|_{1, \Omega_f} + \| (v_2, v_3) \|_{1, \Omega_f} \right]
\]

for \( v \in (\tilde{H}^1(\Omega_f))^3 \times \tilde{H}^1(\Omega_p)^3 \times \tilde{H}^1(\Omega_p)^3 \cap V \) and that
\[
\inf_{\varphi \in V_h} \| u - \varphi \|_V \leq c h \left[ \| u_1 \|_{1, \Omega_f} + \| \nabla \cdot u_1 \|_{1, \Omega_f} + \| v_2 \|_{2, \Omega_p} + \| v_3 \|_{1, \Omega_p} + \| \nabla \cdot v_3 \|_{1, \Omega_p} \right]
\]

(4.3)

for \( v \in (\tilde{H}^1(\Omega_f))^3 \times \tilde{H}^2(\Omega_p)^3 \times \tilde{H}^2(\Omega_p)^3 \cap V \) such that \( \nabla \cdot v_1 \in H^1(\Omega_f) \) and \( \nabla \cdot v_3 \in H^1(\Omega_p) \).

Let \( L \) be a positive integer, \( \Delta t = T/L \), and \( U^n = U(n \Delta t) \). Set
\[
\begin{align*}
&d_r U^n = (U^{n+1} - U^n)/\Delta t, \quad \partial U^n = (U^{n+1} - U^{n-1})/2 \Delta t, \\
&\partial^2 U^n = (U^{n+1} - 2U^n + U^{n-1})/(\Delta t)^2.
\end{align*}
\]

Since we want to use an explicit procedure, we shall compute all integrals involving time-derivative terms using the quadrature rule
\[
\int_Q f(r, z) r \, dr \, d\theta \, dz \approx \frac{2\pi}{4} h_r h_z [f_1 r_1 + f_2 r_2 + f_3 r_3 + f_4 r_4],
\]

(4.5)

where \( f_i \) denotes the value of \( f \) at the node \( a_i \) in the rectangle \( Q \) (see Fig. 2). Note that the rule (4.5) is exact if \( f(r, z) \) is bilinear.
For the elements in \( \mathcal{M}_h \), the rule (4.5) is the natural choice since the local degrees of freedom for any element \( v = (u_r, 0, u_z) \) in \( \mathcal{M}_h \) are the values of \( v \) at the nodes \( a_i, 1 \leq i \leq 4 \). On the other hand, since the local degrees of freedom of a mixed Morley element \( v = (v_r, 0, v_z) \) are the values of \( v \cdot v_Q \) at the midpoints of each side of \( Q \) (that is, the values of \( v \) at the nodes \( a_5 \) and \( a_7 \) and of \( v_z \) at the nodes \( a_6 \) and \( a_8 \)), such values being constant along the sides of \( Q \), the mass-lumping quadrature rule (4.5) can be used for those elements as well.

Let \([v, w]_r\) and \( \|v\|_{0, \Omega, r} = f, p \), denote the inner product \( (v, w) \), and the norm \( \|v\|_{0, \Omega} \) computed approximately using the quadrature rule (4.5). Also, let \( \langle v, w \rangle_r \) denote the inner product \( \langle v, w \rangle_r \) computed using (4.5).

The discrete-time explicit Galerkin procedure is defined as follows. Find \( U^n \in \mathcal{V}_h, n = 0, \ldots, L \), such that

\[
\begin{align*}
[f_r^n, v_1]_r &= \{\varepsilon \varepsilon (U_2, U_3)^n, (v_2, v_3)\}_p
+ \Lambda(U^n, v) + \langle (\rho_f A_f) \partial U_1^n \cdot v_f, v_1 \cdot v_f \rangle_{r_1}
+ \langle B(\partial U_2^n \cdot v_p, \partial U_2^n \cdot \chi_p, \partial U_3^n \cdot v_p), (v_2 \cdot v_p, v_2 \cdot \chi_p, v_3 \cdot v_p) \rangle_{r_1}
+ \langle m \partial U_3^n \cdot v_p, v_3 \cdot v_p \rangle_{r_1}
\end{align*}
\]

(4.6)

We shall analyse the stability of the scheme (4.6). The choice of the test function \( v = \partial U^n \) in (4.6) gives us the inequality

\[
\frac{1}{2\Delta t} \{ \|\rho\|_n d, U_1^{n} \|_{0, \Omega}^2 - \|\rho\|_n d, U_1^{n-1} \|_{0, \Omega}^2
+ \|\varepsilon \varepsilon (U_2, U_3)^n, (v_2, v_3)\|_p - \|\varepsilon \varepsilon (U_2, U_3)^{n-1}, (v_2, v_3)\|_p
+ \langle (\rho_f A_f) \partial U_1^n \cdot v_f, v_1 \cdot v_f \rangle_{r_1}
+ \langle B(\partial U_2^n \cdot v_p, \partial U_2^n \cdot \chi_p, \partial U_3^n \cdot v_p), (v_2 \cdot v_p, v_2 \cdot \chi_p, v_3 \cdot v_p) \rangle_{r_1}
+ \langle m \partial U_3^n \cdot v_p, v_3 \cdot v_p \rangle_{r_1}
\leq C(\|f_r^n\|_{0, \Omega} + \|f_r^n\|_{0, \Omega}) + ||d, U_1^n\|_{0, \Omega} + ||d, U_1^{n-1}\|_{0, \Omega} + ||d, U_1^n\|_{0, \Omega} + ||d, U_1^{n-1}\|_{0, \Omega}. \]

(4.7)

Next, note that

\[
2\Delta t \Lambda(U^n, \partial U^n) = \frac{1}{2} \{ \Lambda(U^{n+1}, U^{n+1}) - \Lambda(U^{n-1}, U^{n-1}) + \Lambda(U^n - U^{n-1}, U^n - U^{n-1}) - \Lambda(U^{n+1} - U^n, U^{n+1} - U^n) \}.
\]

Then, add

\[
\begin{align*}
\frac{\gamma}{4\Delta t} \{ ||U_1^{n+1}\|_{0, \Omega}^2 - ||U_1^{n-1}\|_{0, \Omega}^2
+ ||d, U_1^n\|_{0, \Omega}^2 + ||d, U_1^{n-1}\|_{0, \Omega}^2 + ||U_1^{n+1}\|_{0, \Omega}^2 + ||U_1^{n-1}\|_{0, \Omega}^2
\end{align*}
\]

(4.7)
and

\[
\frac{\gamma}{4\Delta t} \left\{ \| (U_2, U_3)^{n+1} \|_{\Omega_{\beta}}^2 - \| (U_2, U_3)^{n} \|_{\Omega_{\beta}}^2 \right\} \\
\leq \frac{1}{2} \gamma \left( \| d_i(U_2, U_3)^{n} \|_{\Omega_{\beta}}^2 + \| d_i(U_2, U_3)^{n-1} \|_{\Omega_{\beta}}^2 \right) \\
+ \| (U_2, U_3)^{n+1} \|_{\Omega_{\beta}}^2 + \| (U_2, U_3)^{n} \|_{\Omega_{\beta}}^2 \right\}
\]

(4.7)

to multiply by 2\Delta t, and sum the resulting inequality from \( n = 1 \) to \( n = N \), \( 1 \leq N \leq L - 1 \). Since \( \mathbf{C} \) is a non-negative matrix and all the boundary terms in the left-hand side of (4.7) are non-negative,

\[
\| \rho_1^\Delta d_i U_N^N \|_{\Omega_{\beta}}^2 + \| d_i U_N^N \|_{\Omega_{\beta}}^2 - \frac{1}{2} (\Delta t)^2 \mathcal{A}(d_i U_N^N, d_i U^N) \\
+ \frac{1}{2} \left\{ \mathcal{A}_p(U_N^N, U^N) \right\}
\]

(4.8)

Next, note that

\[
\mathcal{A}(d_i U^N, d_i U^N) \leq A_f^* \| \nabla \cdot d_i U_i^N \|_{\Omega_{\beta}}^2 + \lambda_{\text{max}}(E_p) \left( \| \mathcal{E}_{\text{tr}}(d_i U^N) \|_{\Omega_{\beta}}^2 \right) \\
+ \| \mathcal{E}_{\text{ee}}(d_i U^N) \|_{\Omega_{\beta}}^2 + \| \mathcal{E}_{\text{zz}}(d_i U^N) \|_{\Omega_{\beta}}^2 \\
+ \| \mathcal{E}_{\text{tr}}(d_i U^N) \|_{\Omega_{\beta}}^2 + \| \nabla \cdot d_i U_i^N \|_{\Omega_{\beta}}^2.
\]

(4.9)

Also note that there exists a constant \( c_3 \) independent of \( h \) such that the following inverse hypotheses hold:

\[
\begin{aligned}
& (i) \quad \| \nabla \cdot \mathbf{v} \|_{\Omega_{\beta}}^2 \leq c_3 h^{-1} \| \mathbf{v} \|_{\Omega_{\beta}}^2, \quad \mathbf{v} \in W_h(\Omega_i), \quad i = f, p, \\
& (ii) \quad \{ \| \mathcal{E}_{\text{tr}}(\mathbf{v}) \|_{\Omega_{\beta}}^2 + \| \mathcal{E}_{\text{ee}}(\mathbf{v}) \|_{\Omega_{\beta}}^2 + \| \mathcal{E}_{\text{zz}}(\mathbf{v}) \|_{\Omega_{\beta}}^2 \}^{1/2} \\
& \leq c_3 h^{-1} \| \mathbf{v} \|_{\Omega_{\beta}}^2, \quad \mathbf{v} \in M_h.
\end{aligned}
\]

(4.10)

For a uniform grid, a calculation shows that \( c_3 \) is not greater than 6-37; this may not be the best possible constant. In the general case \( c_3 \) will contain a factor that measures the quasiuniformity of the grid.

Then, since \( \lambda_{\text{min}}(\mathcal{A}) > 0 \) (cf. (2.6)), it follows from (4.9) and (4.10) that

\[
\| \rho_1^\Delta d_i U_i^N \|_{\Omega_{\beta}}^2 + \| d_i U_i^N \|_{\Omega_{\beta}}^2 - \frac{1}{2} (\Delta t)^2 \mathcal{A}(d_i U_i^N, d_i U^N) \\
\geq \left( \rho_{f^*} - \left( \frac{\Delta \tau}{h} \right)^2 \frac{C_3}{2} A_f^* \right) \| d_i U_i^N \|_{\Omega_{\beta}}^2 \\
+ \left( \lambda_{\text{min}}(\mathcal{A}) - \left( \frac{\Delta \tau}{h} \right)^2 \frac{C_3}{2} \lambda_{\text{max}}(E_p) \right) \| d_i U_i^N \|_{\Omega_{\beta}}^2 \\
\geq \frac{1}{2} \rho_{f^*} \| d_i U_i^N \|_{\Omega_{\beta}}^2 + \frac{1}{2} \lambda_{\text{min}}(\mathcal{A}) \| d_i U_i^N \|_{\Omega_{\beta}}^2,
\]

(4.11)
where $\rho_f$ is the minimum of $\rho_f(r, z)$ in $\Omega_f$ and provided that $\Delta t$ and $h$ satisfy the stability condition

$$\Delta t \leq \frac{h}{c_3 \min \left( \left( \frac{\rho_f}{A_f} \right)^{\frac{1}{3}}, \left( \frac{\lambda_{\min}(\mathbf{A})}{\lambda_{\max}(E_p)} \right)^{\frac{1}{3}} \right)}.$$ \hfill (4.12)

(A modification of the argument above would permit us to replace the term $(\lambda_{\min}(\mathbf{A})/\lambda_{\max}(E_p))^{\frac{1}{3}}$ by the reciprocal of the maximum wave velocity in $\Omega_p$; the constant $c_3$ may be different in this case.)

Also, note that for $\Delta t$ and $h$ as in (4.12),

$$\frac{1}{2}(\Delta t)^2 A(d, U^n, d, U^n) \leq C[\|d, U^n\|_{0, \mathcal{Q}}^2 + \|d, (U_2, U_3)^0\|_{0, \mathcal{Q}}^2]. \hfill (4.13)$$

Next, an easy calculation shows that there exists a constant $c_4$ independent of $h$ such that

$$\|v_1\|_{0, \mathcal{Q}} \leq c_4 \|v_1\|_{0, \mathcal{Q}}$$

and

$$\|(v_2, v_3)\|_{0, \mathcal{Q}} \leq c_4 \|(v_2, v_3)\|_{0, \mathcal{Q}}$$

for any $v \in V_A$. Thus, using (4.11), (4.12), (4.13), and the $V$-coercivity of $A_v$ in (4.8), we see the following inequality holds:

\[
\begin{align*}
\|d, U^n\|_{0, \mathcal{Q}}^2 + \|d, (U_2, U_3)^0\|_{0, \mathcal{Q}}^2 &\leq C \left( \|d, U^n\|_{0, \mathcal{Q}}^2 + \|d, (U_2, U_3)^0\|_{0, \mathcal{Q}}^2 + \|U^n\|_{\mathcal{V}} + \|U^n\|_{\mathcal{V}}^2 \\
&\quad + \|d, (U_2, U_3)^0\|_{0, \mathcal{Q}}^2 + \|d, (U_2, U_3)^0\|_{0, \mathcal{Q}}^2 + \|d, (U_2, U_3)^0\|_{0, \mathcal{Q}}^2 \\
&\quad + \|d, (U_2, U_3)^0\|_{0, \mathcal{Q}}^2 + \|U^n\|_{\mathcal{V}} + \|U^n\|_{\mathcal{V}}^2 \right). \quad 1 \leq N \leq L - 1.
\end{align*}
\]

Then Gronwall's lemma implies that

\[
\max_{1 \leq N \leq L - 1} \left( \|d, U^n\|_{0, \mathcal{Q}} + \|d, (U_2, U_3)^0\|_{0, \mathcal{Q}} + \|U^n\|_{\mathcal{V}} \right)
\]

\[
\leq C \left( \|d, U^n\|_{0, \mathcal{Q}} + \|d, (U_2 - U_3)^0\|_{0, \mathcal{Q}} + \|U^0\|_{\mathcal{V}} + \|U^n\|_{\mathcal{V}} + \|d, (U_2 - U_3)^0\|_{0, \mathcal{Q}}^2 + \|d, (U_2 - U_3)^0\|_{0, \mathcal{Q}}^2 + \|U^n\|_{\mathcal{V}} + \|U^n\|_{\mathcal{V}}^2 \right), \quad \text{ (4.14)}
\]

which shows that the scheme (4.6) is stable under the condition (4.12). It is also obvious that (4.14) gives us existence and uniqueness for the solution $(U^n)_{1 \leq n \leq L - 1}$ of (4.6).

Finally, since the quadrature rule employed in the procedure is $O(h^2)$-correct, a standard argument combining the approximating properties (4.3) and (4.4) with the ideas leading to (4.14) would give us the optimal order error estimate

\[
\max_{1 \leq N \leq L - 1} \left( \|d, (u_1 - U_1)^0\|_{0, \mathcal{Q}} + \|d, (u_2 - U_2, u_3 - U_3)^0\|_{0, \mathcal{Q}} + \|(u - U)^0\|_{\mathcal{V}} \right)
\]

\[
\leq C(u) \left[ \|d, (u_1 - U_1)^0\|_{0, \mathcal{Q}} + \|d, (u_2 - U_2, u_3 - U_3)^0\|_{0, \mathcal{Q}} + \|(u - U)^0\|_{\mathcal{V}} + \|(u - U)^1\|_{\mathcal{V}} + (\Delta t)^2 + h \right].
\]
5. Derivation of the absorbing boundary conditions

In this section we shall derive the absorbing boundary condition (2.8.ii) for the artificial boundary $F_2$ of $Q_p$. In this derivation we shall use some results of [9] as well as some ideas given in [11] for obtaining absorbing boundary conditions for anisotropic elastic solids.

Let us consider a wave front arriving normally to $F_2$ with velocity $c$. Following [8], the strain tensor $\varepsilon(u_5^c)$ on $F_2$ can be written in the form

$$\varepsilon_r(u_5^c) = -\frac{1}{c} \frac{\partial u_5^c}{\partial t} v_{pr}, \quad \varepsilon_{\theta \theta}(u_5^c) = \frac{u_5^c}{r} = 0, \quad \varepsilon_{zz}(u_5^c) = -\frac{1}{c} \frac{\partial u_5^c}{\partial t} v_{pz},$$

$$\varepsilon_z(u_5^c) = -\frac{1}{2c} \left( \frac{\partial u_5^c}{\partial t} v_{pz} + \frac{\partial u_5^c}{\partial l} v_{pr} \right), \quad (r, \theta, z) \in F_2, t \in J. \quad (5.1)$$

In particular,

$$\nabla \cdot u_5^c = -\frac{1}{c} \frac{\partial u_5^c}{\partial t} \cdot v_p. \quad (5.2)$$

Also,

$$\nabla \cdot u_5^c = -\frac{1}{c} \frac{\partial u_5^c}{\partial t} \cdot v_p. \quad (5.3)$$

Next, let us introduce the variables

$$\alpha_1^c = \frac{1}{c} \frac{\partial u_5^c}{\partial t} \cdot v_p, \quad \alpha_2^c = \frac{1}{c} \frac{\partial u_5^c}{\partial t} \cdot \chi_p, \quad \alpha_3^c = \frac{1}{c} \frac{\partial u_5^c}{\partial t} \cdot \chi_p^2, \quad \alpha_4^c = \frac{1}{c} \frac{\partial u_5^c}{\partial t} \cdot v_p,$$

and set

$$\alpha^c = (\alpha_1^c, \alpha_2^c, \alpha_3^c, \alpha_4^c)^T.$$

Combining the stress–strain relations (2.1) with (5.1), (5.2), and (5.3) shows that the strain-energy density $W_p$ on $F_2$ can be written as a quadratic function $\Pi(\alpha^c) = W_p(\varepsilon(\alpha^c), \nabla \cdot u_5(\alpha^c))$ in the form

$$\Pi(\alpha^c) = \frac{1}{2} (\alpha^c)^T \tilde{E}_p \alpha^c, \quad (5.4)$$

where $\tilde{E}_p \in \mathbb{R}^{4 \times 4}$ is the symmetric, positive-definite matrix given by

$$\tilde{E}_p = \begin{pmatrix} A + 2N & 0 & 0 & Q \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ Q & 0 & 0 & H \end{pmatrix}.$$

Next, note that the momentum equations on $F_2$ are given by

$$c \mathcal{A} \frac{\partial (u_5^c, u_3)}{\partial t} = \left( - \sum_{i,j=r,\theta,z} \frac{\partial W_p}{\partial \varepsilon_{ij}} v_{p_i} \frac{\partial W_p}{\partial \nabla \cdot u_3} v_p \right)$$
for \((r, \theta, z) \in \Omega_2, t \in J\). Alternatively, they can be written in the form (cf. (2.4))

\[
\begin{align*}
(i) & \quad c[\rho \partial u^z_\tau / \partial t + \rho_f \partial u^r_\tau / \partial t] = -\tau v_p, \\
(ii) & \quad c[\rho_f \partial u^z_\tau / \partial t + g \partial u^r_\tau / \partial t] = p v_p, \quad (r, \theta, z) \in \Omega_2, t \in J.
\end{align*}
\]

Now, we shall write equations (5.5) in terms of the new variables \(\alpha^c, 1 \leq i \leq 4\).

First, note that taking the inner product of (5.5.ii) with the tangent vectors \(\chi^m_p, m = 1, 2\), gives the relations

\[
\frac{\partial u^z_\tau}{\partial t} \cdot \chi^m_p = -g^{-1} \rho_f \frac{\partial u^r_\tau}{\partial t} \cdot \chi^m_p, \quad m = 1, 2.
\]

Let \(\mathcal{E} = (\tau v_p \cdot v_p, \tau v_p \cdot \chi^1_p, \tau v_p \cdot \chi^2_p, -p)^T\). Then, take the inner product of (5.5.i) with \(v_p^m, m = 1, 2\), and of (5.5.ii) with \(v_p^m\) to get the equations

\[
c^2 \tilde{\mathcal{A}} \alpha^c = -\mathcal{E} = \frac{\partial \Pi}{\partial \alpha^c} = \mathcal{E}^* p \alpha^c,
\]

\(\tilde{\mathcal{A}} \in \mathbb{R}^{4 \times 4}\) being the symmetric, positive-definite matrix defined by

\[
\tilde{\mathcal{A}} = \begin{pmatrix}
\rho & 0 & 0 & \rho_f \\
0 & \rho - g^{-1}(\rho_f)^2 & 0 & 0 \\
0 & 0 & \rho - g^{-1}(\rho_f)^2 & 0 \\
\rho_f & 0 & 0 & g
\end{pmatrix}
\]

Let \(S = \tilde{\mathcal{A}}^{-1} \mathcal{E}^* p \tilde{\mathcal{A}}^{-1}, \tilde{\alpha}^c = \tilde{\mathcal{A}}^{1/2} \alpha^c\). Then equation (5.6) becomes

\[
S \tilde{\alpha}^c = c^2 \tilde{\alpha}^c.
\]

Also, in terms of \(\tilde{\alpha}^c\) the strain-energy density on \(\Omega_2\) can be written in the form

\[
\tilde{\pi}(\tilde{\alpha}^c) = \pi(\alpha^c) = \frac{1}{2}(\tilde{\alpha}^c)^T S \tilde{\alpha}^c.
\]

Let \(c_i, 1 \leq i \leq 4\), be the four positive wave velocities satisfying (5.7); that is, solutions of the equation

\[
\det (S - c^2 I) = 0.
\]

Two of these roots are

\[
c_2 = c_3 = \left(\frac{N}{\rho - g^{-1}(\rho_f)^2}\right)^{\frac{1}{2}}
\]

and they correspond to the shear modes of propagation. The other two velocities \(c_1\) and \(c_4\) are distinct and they correspond to the compressional modes of propagation and have a more complicated expression in terms of the mass and stiffness coefficients of \(Q_p\). It can be easily checked that these values coincide with the corresponding ones obtained in [1] by Biot using a different argument.

Let \(M_i, 1 \leq i \leq 4\), be the set of orthonormal eigenvectors corresponding to \(c_i, 1 \leq i \leq 4\), and let \(M\) be the matrix containing the eigenvectors \(M_i\) of \(S\) as rows, and \(D\) be the diagonal matrix containing the eigenvalues \(c_i^2, 1 \leq i \leq 4\), of \(S\), so that \(S = M^T D M\).
Let
\[ \alpha = \left( \frac{\partial u_2}{\partial t}, \frac{\partial u_2}{\partial t} \cdot \chi_p, \frac{\partial u_2}{\partial t} \cdot \chi_p, \frac{\partial u_2}{\partial t} \cdot \chi_p, \frac{\partial u_3}{\partial t} \cdot \nu \right) \]
be a general velocity on the surface \( F_2 \) due to the simultaneous normal arrival of waves of velocities \( c_i, \ 1 \leq i \leq 4 \). Since the \( M_i \) are orthonormal, we can write \( \bar{\alpha} = \mathbf{A}^1 \alpha \) in the form
\[ \bar{\alpha} = \sum_{i=1}^{4} [M_i, \mathbf{A}^1 \alpha]_e M_i. \]

Let
\[ \bar{\alpha}^i = \mathbf{A}^1 \alpha^i = \frac{1}{c_i} [M_i, \mathbf{A}^1 \alpha]_e M_i, \quad 1 \leq i \leq 4. \] (5.9)

Then, \( \bar{\alpha}^i \) satisfies
\[ S \bar{\alpha}^i = c_i^2 \bar{\alpha}^i, \] (5.10)
and
\[ \pi(\bar{\alpha}^i) = \frac{1}{2} (\bar{\alpha}^i)^T S \bar{\alpha}^i. \] (5.11)

Using (5.6) and (5.11), we see that the force \( \mathcal{F}_i \) on \( F_2 \) corresponding to \( \bar{\alpha}^i \) satisfies the relations
\[ \mathbf{A}^1 \frac{\partial \bar{\pi}}{\partial \bar{\alpha}^i} = \mathbf{A}^1 S \bar{\alpha}^i = \mathbf{E}_p \alpha^i = -\mathcal{F}_i. \] (5.12)

We observe that the interaction energy among the different types of waves arriving normally to \( F_2 \) is small compared to the total energy involved [5]. Thus, neglecting such interaction, we can write the total strain energy density
\[ \bar{\pi} = \sum_{i=1}^{4} \bar{\pi}(\bar{\alpha}^i) \]
as the sum of the partial energies and the total force \( \mathcal{F} \) on \( F_2 \) as the sum of the forces corresponding to each \( \bar{\alpha}^i \), so that, according to (5.12),
\[ \mathcal{F} = \sum_{i=1}^{4} \mathcal{F}_i = -\mathbf{A}^{-1} \sum_{i=1}^{4} S \bar{\alpha}^i. \]

Since we can write \( \mathbf{A}^{-1} \mathcal{F} \) in the form
\[ \mathbf{A}^{-1} \mathcal{F} = \sum_{i=1}^{4} [M_i, \mathbf{A}^{-1} \mathcal{F}]_e M_i, \]
it follows that
\[ S \bar{\alpha}^i = -[M_i, \mathbf{A}^{-1} \mathcal{F}]_e M_i, \quad 1 \leq i \leq 4. \] (5.13)
Thus, combining (5.9), (5.10) and (5.13), we see that
\[ c_i^2 \bar{\alpha}^i = c_i [M_i, \mathbf{A}^1 \alpha]_e M_i = S \bar{\alpha}^i = -[M_i, \mathbf{A}^{-1} \mathcal{F}]_e M_i, \quad 1 \leq i \leq 4, \]
so that
\[ c[M, A^1 \alpha] = -[M, A^{-1} \mathcal{F}], \quad 1 \leq i \leq 4. \]

In matrix form the equation above becomes
\[-M A^{-1} \mathcal{F} = D^1 M A^1 \alpha,\]
so that after multiplying by \( A^1 M^\dagger \) we obtain the relations
\[-\mathcal{F} = A^1 S^1 A^1 \alpha = [(A^{-1} E)^\dagger] A^1 \alpha = B \alpha.\]

These are the equations used as boundary conditions for the artificial boundary \( F_2 \). Note that the matrix \( B \) is symmetric and positive definite.

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**References**


