GEOPHYSICS, VOL. 52, NO. 2 (FEBRUARY 1987); P. 174–178, 2 FIGS., 2 TABLES.

Boundary conditions for a fluid-saturated porous solid

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ABSTRACT

Using Biot's theory, a set of boundary conditions is presented for wave transmission and reflection at the contact surface between an elastic medium (fluid or solid) and a fluid-saturated porous solid (Biot medium).

The analysis shows the continuity of the normal component of the density energy flux vectors across the contact surfaces, so that total energy is preserved.

Energy reflection and transmission coefficients are computed for each kind of Biot wave.

INTRODUCTION

The theory of consolidation and wave propagation in a porous elastic solid containing a compressible fluid has been stated by Biot in several articles (Biot, 1941, 1956, 1962; Biot and Willis, 1957). Biot and Willis (1957) discuss methods for measuring the coefficients characterizing the elastic properties for a porous elastic solid. They also discuss the physical significance of the coefficients.

Dutta and Odé (1983) have shown that viscous fluid flow affects seismic wave amplitudes and reflection coefficients at a gas-water contact between two Biot media. They have also posed the boundary-value problem for a particular case in which a Biot wave of the first kind hits a planar interface, and both media are handled as isotropic, fluid-saturated porous media.

However, the formulation and analysis of direct and inverse problems in anisotropic and inhomogeneous Biot media with contact surfaces of arbitrary shapes call for the derivation of more general boundary conditions applicable to each interface. By using the stress-strain relations for isotropic, fluid-saturated porous media, general boundary conditions are developed here for the cases in which the media in contact are (1) two fluid-saturated porous solids; (2) a fluid-saturated porous solid and a solid; or (3) a fluid-saturated porous solid and a fluid.

The continuity of the normal component of the density energy flux vectors is maintained in applying these boundary conditions, so that the total energy is preserved.

Finally, the density energy flux vectors are used to compute the energy transmission and reflection coefficients for the three cases mentioned above. The expressions derived for these coefficients, which are useful for quantitative estimations, generalize those obtained by Dutta and Odé (1983) for a gas-water contact.

STRESS-STRAIN RELATIONS OF A BIOT MEDIUM

Let $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ denote the average displacement vector, where $\mathbf{u}_1 = (u_{11}, u_{12}, u_{13})$ and $\mathbf{u}_2 = (u_{21}, u_{22}, u_{23})$ are the displacements in the solid and fluid parts of the aggregate, respectively.

Einsteinian notation (in which repeated indices are summed) is used hereafter. In this notation, the solid and fluid strain components are given by

$$e_{ij} = (1/2)(\partial u_{1i}/\partial x_j + \partial u_{1j}/\partial x_i),$$

$$e = \partial u_{1i}/\partial x_i,$$

and

$$\varepsilon = \partial u_{2i}/\partial x_i.$$

The stress-strain relations for the isotropic case are

$$\theta_{ij} = 2Ne_{ij} + (Ae + Q\varepsilon)\delta_{ij},$$

and (1)

$$S = Qe + R\varepsilon,$$

where δ_{ij} is the Kronecker delta.

On a unit cube of the Biot medium, θ_{ij} and S represent the forces acting on the solid and fluid portions of each side of the cube, respectively. The scalar S is related to the fluid pressure p according to

$$S = -\beta p, \tag{2}$$

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where β is the effective porosity. The effective porosity represents the interconnecting void space. Here the sealed pore space is considered part of the solid.

Following Biot and Willis (1957), let k and ρ be the jacketed and unjacketed compressibilities, respectively. Also let ψ denote the fluid content referred to in the unjacketed compressibility test. Then, the elastic coefficients A, N, Q, and R can be expressed as

$$A = \left[\psi/k + \beta^2 + (1 - 2\beta)(1 - \rho/k)\right] D^{-1} - (2/3)\mu,$$

$$N = \mu,$$

$$Q = \beta(1 - \beta - \rho/k)D^{-1},$$

and

$$R = \beta^2 D^{-1},$$

where

$$D = \psi + \rho - \rho^2/k.$$

Since most exploration geophysicists are more familiar with Gassmann's notation (Gassmann, 1951) than with the notation used by Biot, the relations between the elastic coefficients above and Gassmann's parameters (matrix compressibility c_m, bulk compressibility c_b, and fluid compressibility c_f) are given below (Geertsma and Smit, 1961):

$$A = (1 - c_m/c_b - \beta)^2 M + 1/c_b - (2/3)\mu,$$

$$Q = (1 - c_m/c_b - \beta)\beta M,$$

and

$$R = \beta^2 M,$$

where

$$M = \left[(1 - c_m/c_b - \beta)c_m + c_f\right]^{-1}.$$

ANALYSIS OF THE BOUNDARY CONDITIONS

A discontinuity surface is defined as a surface where, owing to a sharp change in porosity, fluid, or both, a discontinuity occurs in the elastic coefficients describing the media. Examples where the discontinuity surface is well defined are (1) a porous medium containing two nonmiscible fluids, and (2) a single fluid saturating a porous medium with a great change in porosity along a given surface.

Provided that the discontinuity surface is well defined, one can state its boundary conditions. Let the relative displacement vector $\mathbf{w} = (w_1, w_2, w_3)$ be defined by

$$\mathbf{w} = \beta(\mathbf{u}_2 - \mathbf{u}_1). \tag{4}$$

The vector $\mathbf{w}$ represents the flow of the fluid relative to the solid, measured in terms of volume per unit area of the bulk medium. In other words, $\mathbf{w}$ is the difference between the average fluid displacement on each face of a unit cube of aggregate and the corresponding displacement for a solid unit cube of the same material.

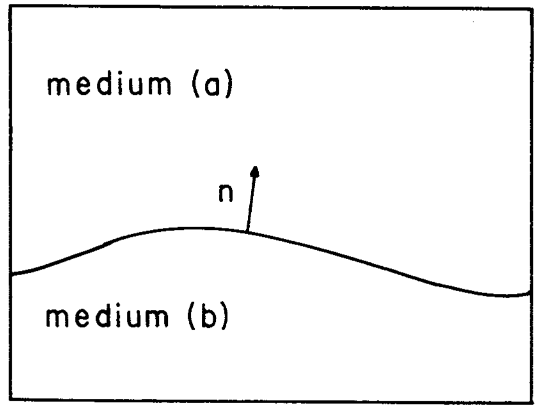

FIG. 1. Discontinuity surface between media a and b.

Case 1—Two Biot media in contact

The natural boundary conditions across the discontinuity surface between two different, nonmiscible fluid-saturated porous solids are given by the following relations:

Continuity of normal stresses, or

$$n_i(\theta_{aij} + S_a\delta_{ij}) = n_i(\theta_{bij} + S_b\delta_{ij}); \tag{5a}$$

continuity of fluid pressure, or

$$S_a/\beta_a = S_b/\beta_b; \tag{5b}$$

continuity of the solid displacement vector, or

$$u_{a1} = u_{b1}; \tag{5c}$$

continuity of the normal component of the relative displacement vector, or

$$\beta_a(u_{a2i} - u_{a1i})n_i = \beta_b(u_{b2i} - u_{b1i})n_i. \tag{5d}$$

Here, subscripts a and b indicate the properties of the two media, and $\mathbf{n} = (n_1, n_2, n_3)$ is the unit outward normal of medium b along the discontinuity surface (Figure 1).

Analysis of the behavior of medium b when porosity tends to zero or one (i.e., elastic solid or fluid) allows us to derive the boundary conditions for the remaining cases from the corresponding conditions in expressions (5) of case 1. It is convenient to rewrite equations (5a) and (5b) in an equivalent form. Note that, regardless of medium b, the fluid in medium a must always satisfy the condition $S_a/\beta_a = n_i\tau_{ij}n_j$, where $\tau_{ij} = \theta_{ij} + S\delta_{ij}$ is the total force per unit area of the medium. Since for case 1 the last equation also holds for medium b, then $S_b/\beta_b = n_i\tau_{ij}n_j$. Using equation (5a), it follows that $S_a/\beta_a = S_b/\beta_b$, so that the continuity in fluid pressure is recovered. Hence equations (5a) and (5b) can be replaced by

$$n_i(\tau_{aij} - \tau_{bij}) = 0, \tag{6a}$$

and

$$\begin{aligned} S_a &= \beta_a\tau_{bij}n_in_j, \\ S_b &= \beta_b\tau_{aij}n_in_j. \end{aligned} \tag{6b}$$

If the porosity in medium b tends to zero, it follows from equations (3) that Q_b, R_b, and ψ_b tend to zero. Also note that for the limit where $\beta = 0$, both parameters k_b and ρ_b coincide with the solid compressibility k_s. This is because in both the jacketed and unjacketed compressibility tests the entire pressure is transmitted to the solid (Biot and Willis, 1957).

Table 1. Boundary conditions for cases 1, 2, and 3.

(1) Two Biot media	(2) Biot medium-Solid	(3) Biot medium-Fluid
$n_i\left[(\theta_{aij} + S_a\delta_{ij}) - (\theta_{bij} + S_b\delta_{ij})\right] = 0$	$n_i\left[(\theta_{aij} + S_a\delta_{ij}) - \Sigma_{bij}\right] = 0$	$n_i\left[(\theta_{aij} + S_a\delta_{ij}) - S_b\delta_{ij}\right] = 0$
$S_a/\beta_a = S_b/\beta_b$	$S_a/\beta_a = \Sigma_{bij}n_i n_j$	$S_a/\beta_a = S_b$
$u_{a1} = u_{b1}$	$u_{a1} = u_{b1}$	$n_i\left[\beta_a u_{a2i} + (1-\beta_a)u_{a1i} - u_{b2i}\right] = 0$
$n_i\left[(u_{a2i} - u_{a1i})\beta_a - (u_{b2i} - u_{b1i})\beta_b\right] = 0$	$n_i(u_{a2i} - u_{a1i})\beta_a = 0$	

Writing expression (3) as

$$A = (1/k)\left[\psi + k(1 - \rho/k)\right]D^{-1} + \left[\beta^2 - 2\beta(1-\rho/k)\right]D^{-1} - (2/3)\mu,$$

it can be easily seen that

$$A \xrightarrow[\beta\to 0]{} 1/k - (2/3)\mu = \lambda, \tag{7}$$

where λ is the Lamé solid coefficient.

I next discuss the behavior of the elastic parameters when porosity tends to one. First note that the fluid content ψ tends to the fluid compressibility ρ_f and ρ tends to zero. Furthermore, it can be seen that the jacketed compressibility $k = -e/p$ tends to infinity, since strain increases indefinitely for a fixed value of pressure p (Biot and Willis, 1957). From the foregoing, the following limit values can be obtained:

$$A_b \xrightarrow[\beta_b\to 1]{} 0,$$

$$Q_b \xrightarrow[\beta_b\to 1]{} 0, \tag{8}$$

and

$$R_b \xrightarrow[\beta_b\to 1]{} \rho_f^{-1}.$$

Having performed the analysis of the asymptotic behavior of medium b, I now derive the boundary conditions for the other two cases.

Case 2—Fluid-saturated porous solid-solid interface

The boundary conditions to be satisfied at the interface between the media will be derived by taking limits when the porosity β_b tends to zero in expressions (5). Let Σ_b be the solid stress tensor. Then considering the asymptotic behavior of the elastic parameters, it follows that

$$\theta_b \xrightarrow[\beta_b\to 0]{} \Sigma_b,$$

$$S_b \xrightarrow[\beta_b\to 0]{} 0, \tag{9}$$

and

$$\beta_b(u_{b2i} - u_{b1i}) \xrightarrow[\beta_b\to 0]{} 0.$$

Thus, the boundary conditions are given by the relations

$$n_i(\theta_{aij} + S_a\delta_{ij} - \Sigma_{bij}) = 0, \tag{10a}$$

$$S_a = \beta_a\Sigma_{bij}n_i n_j, \tag{10b}$$

$$u_{a1} = u_{b1}, \tag{10c}$$

and

$$n_i\beta_a(u_{a2i} - u_{a1i}) = 0. \tag{10d}$$

Note that equation (10a) expresses the continuity of total stress at the "discontinuity surface," whereas equation (10b) provides the behavior of fluid pressures. Also equation (10c) is just the continuity of solid displacements, and equation (10d) states that the normal component of the relative flow along the discontinuity surface is zero, which means that the solid medium b is impervious and the saturation of medium a is preserved.

Case 3—Fluid-saturated porous solid-fluid interface

First note that from equation (5c), $n_i u_{a1i} = n_i u_{b1i}$, so that substitution in equation (5d) yields

$$\left[\beta_a(u_{a2i} - u_{a1i}) - \beta_b(u_{b2i} - u_{a1i})\right]n_i = 0. \tag{11}$$

As in case 2, by taking limits in equations (5a), (5b), and (11) when the porosity tends to one, the following relations are obtained:

$$n_i\left[(\theta_{aij} + S_a\delta_{ij}) - S_b\delta_{ij}\right] = 0, \tag{12a}$$

$$S_a = \beta_a S_b, \tag{12b}$$

and

$$n_i\left[\beta_a u_{a2i} + (1-\beta_a)u_{a1i}\right] = n_i u_{b2i}. \tag{12c}$$

The first condition in equation (12a) again yields the continuity of total stresses on the surface, although transverse stresses for the solid part (θ_{ij} with $i \neq j$) are zero.

From equation (12b), the pressures on the fluids (i.e., $S_a/\beta_a = S_b/\beta_b$) are shown to be continuous, along with the stress quotient $S_a/S_b = \beta_a$ which is the relation between the fluid area and total area in the saturated porous medium.

The continuity of normal displacements is expressed in equation (12c) and, as expected, no condition on transverse displacements is imposed.

The boundary conditions obtained, which are summarized in Table 1, were derived using the stress-strain relations for isotropic media. However, the tensor form in which these conditions are expressed makes them applicable to cases where the media are anisotropic.

ENERGY FLUX

Recall that the density energy flux vector ϕ (Ben-Menahen and Singh, 1981) is given by

$$\phi_{sj} = \Sigma_{bij}\dot{u}_{b1i} \quad \text{for an elastic solid,}$$

and

$$\phi_{fj} = S_b \dot{u}_{b2j} \quad \text{for a fluid,}$$

where a dot means differentiation with respect to time t. Thus, the generalized density energy flux vector ϕ_p for a Biot medium can be defined as

$$\phi_{pj} = \theta_{ij}\dot{u}_{1i} + S\delta_{ij}\dot{u}_{2i}.$$

Consider a small volume V across a discontinuity surface of negligible transverse areas s_1 and s_2 (Figure 2). For a gas-water contact in a porous solid interface, by applying the boundary conditions and integrating vector ϕ_p over a vanishing volume V, Dutta and Odé (1983) proved that the total energy is conserved.

For cases 2 and 3, the behavior of the density energy flux vector along the interface is analyzed below.

For case 2, using expressions (10),

$$\begin{aligned} n_j\phi_{sj} &= \Sigma_{bij}\dot{u}_{b1i}n_j \\ &= (\Sigma_{aij} + S_a\delta_{ij})\dot{u}_{a1i}n_j \\ &= n_j\phi_{pj}. \end{aligned}$$

Similarly, for case 3, using expressions (12),

$$\begin{aligned} n_j\phi_{fj} &= S_b\dot{u}_{b2i}n_i \\ &= S_b\left[\beta_a\dot{u}_{a2j} + (1-\beta_a)\dot{u}_{a1j}\right]n_j \\ &= S_a\dot{u}_{a2j}n_j + (S_b - S_a)n_j\dot{u}_{a1j} \\ &= S_a\dot{u}_{a2j}n_j + n_i\theta_{aij}\dot{u}_{a1j} \\ &= n_j\phi_{pj}. \end{aligned}$$

Thus, the density energy flux vector is continuous along the discontinuity surface, which in turn implies that the total energy is preserved.

ENERGY REFLECTION AND TRANSMISSION COEFFICIENTS

In order to obtain the energy reflection and transmission coefficients, the work per unit area is defined by

$$\Phi = (1/\Upsilon)\int_0^{\Upsilon} n_i\phi_i\,dt, \qquad i = 1, 2, 3,$$

where Υ is the main period and ϕ is the density energy flux vector, which has already been defined for each case. The superscripts I, p, s, c, and d indicate contributions of incident, compressional, shear, and compressional type I and II Biot waves, respectively, to the field variables or reflection coefficients.

Following Dutta and Odé (1983), for computing the reflection and transmission coefficients the stress and displacement

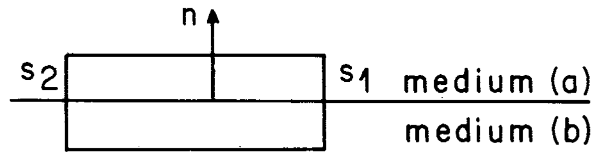

FIG. 2. The location of vanishing volume across the discontinuity surface in computing energy flux.

vectors must be separated into a number of terms depending upon the different kinds of waves involved. As an example, for case 2, when the incident wave comes from the solid part, such splitting can be written as

$$\begin{aligned} \Sigma_b &= \Sigma_b^I + \Sigma_b^p + \Sigma_b^s, \\ S_a &= S_a^c + S_a^d + S_a^s, \\ \theta_a &= \theta_a^c + \theta_a^d + \theta_a^s, \\ \dot{u}_{b1} &= \dot{u}_{b1}^{\mathrm{I}} + \dot{u}_{b1}^{\mathrm{R}}, \\ \dot{u}_{b1}^{\mathrm{R}} &= \dot{u}_{b1}^{p} + \dot{u}_{b1}^{s}, \end{aligned}$$

and

$$\dot{u}_{aj} = \dot{u}_{aj}^c + \dot{u}_{aj}^d + \dot{u}_{aj}^s = \dot{u}_{aj}^{\mathrm{T}}, \qquad j = 1, 2.$$

The superscripts R and T denote the reflection and transmission displacement vectors, respectively. For this example,

$$\Phi^{\mathrm{I}} = (1/\Upsilon)\int_0^{\Upsilon} \Sigma_{bij}\dot{u}_{b1i}^{\mathrm{I}}n_j\,dt,$$

$$\Phi^{\mathrm{R}} = (1/\Upsilon)\int_0^{\Upsilon} \Sigma_{bij}\dot{u}_{b1i}^{\mathrm{R}}n_j\,dt,$$

and

$$\Phi^{\mathrm{T}} = (1/\Upsilon)\int_0^{\Upsilon} (\theta_{aij}\dot{u}_{a1i} + S_a\delta_{ij}\dot{u}_{a2i})n_j\,dt$$

are the incident, reflected, and transmitted energy fluxes. The reflection and transmission coefficients can be defined as the ratios

$$R = \Phi^{\mathrm{R}}/\Phi^{\mathrm{I}}, \qquad T = \Phi^{\mathrm{T}}/\Phi^{\mathrm{I}}.$$

Energy reflection and transmission coefficients for each kind of wave can also be computed. Let the partial fluxes (Φ^{km}) be defined as follows:

$$\Phi^{km} = (1/\Upsilon)\int_0^{\Upsilon} \Sigma_{ij}^k\dot{u}_{1i}^m n_j\,dt,$$

$$k, m = I, p, s \text{ (solid medium);}$$

$$\Phi^{km} = (1/\Upsilon)\int_0^{\Upsilon} S^k\dot{u}_{2j}^m n_j\,dt,$$

$$k, m = I, p \text{ (fluid medium);}$$

$$\Phi^{km} = (1/\Upsilon)\int_0^{\Upsilon} (\theta_{ij}^k\dot{u}_{1i}^m + S^k\dot{u}_{2j}^m)n_j\,dt,$$

$$k, m = I, c, d, s \text{ (Biot medium).}$$

Dutta and Odé (1983) called the functions Φ^{km} "orthodox

Table 2. Energy reflection and transmission coefficients.

Interface	R^p	R^c	R^d	R^s	T^p	T^c	T^d	T^s
Solid-Biot medium	Φ^{pp}			Φ^{ss}		Φ^{cc}	Φ^{dd}	Φ^{ss}
Biot medium-Solid		Φ^{cc}	Φ^{dd}	Φ^{ss}	Φ^{pp}			Φ^{ss}
Fluid-Biot medium	Φ^{pp}					Φ^{cc}	Φ^{dd}	Φ^{ss}
Biot medium-Fluid		Φ^{cc}	Φ^{dd}	Φ^{ss}	Φ^{pp}			
Two Biot media		Φ^{cc}	Φ^{dd}	Φ^{ss}		Φ^{cc}	Φ^{dd}	Φ^{ss}

fluxes" if $k = m$ and "interference fluxes" if $k \neq m$. The orthodox fluxes normalized to incident fluxes represent the energy reflection and transmission coefficients of each wave (i.e., $R^p = \Phi^{pp}/\Phi^{I}$, etc.).

Table 2 summarizes such coefficients for all possible cases where, for simplicity, I have chosen a unitary incident flux ($\Phi^{I} = 1$). The first column in the table indicates the two media in contact and the incident wave always comes from the first medium.

CONCLUSIONS

Biot's theory is applied to the problem of wave propagation through a contact surface involving a fluid-saturated porous solid. Boundary conditions are derived from the stress-strain relations for an isotropic medium taking the asymptotic behavior of the elastic parameters to be functions of the porosity. Contrary to previous works, these conditions are presented as a set of equations imposing the continuity of the stress tensors and the displacement vectors, so that they can be applied at contact surfaces of arbitrary shape and remain invariant for anisotropic media. Furthermore, the reflection and transmission coefficients, which are of great interest in exploration geophysics, can be easily evaluated from the expressions obtained here.

The results presented here are valuable in the formulation and analysis of direct and inverse problems in geophysics, and also in the design of numerical simulators for wave propagation in geologic models, including reservoirs trapped in any type of rocks.

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