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Finite Element Methods for a Model for Full Waveform Acoustic Logging

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Dedicated to Professor Leslie Fox on the occasion of his seventieth birthday

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A model is defined to simulate the propagation of waves in a radially symmetric, isotropic, composite system consisting of a fluid-filled well bore Ω_f through a fluid-saturated porous solid Ω_p. Biot's equations of motion are chosen to describe the propagation of waves in Ω_p, while the standard equation of motion for compressible inviscid fluids is used for Ω_f, with appropriate boundary conditions at the contact surface between Ω_f and Ω_p. Also, absorbing boundary conditions for the artificial boundaries of Ω_p are derived for the model, their effect being to make them transparent for waves arriving normally.

First, results on the existence and uniqueness of the solution of the differential problem are given and then a discrete-time, explicit finite element procedure is defined and analysed, with finite element spaces suited for radially symmetric problems being used for the spatial discretisation.

1. Introduction

WE CONSIDER the problem of acoustic and elastic wave propagation in a cylindrical fluid-filled borehole Ω_f through a fluid-saturated porous solid Ω_p. The problem arises naturally in acoustic well-logging. A compressional point source is excited at a point on the centreline of the borehole, and the energy transmitted through the fluid in the borehole and through the surrounding formations is recorded by receivers located in the well bore, both above and below the source, to obtain what is known as a full waveform acoustic log [19]. Here, in order to simplify the problem we have assumed that the whole system $\Omega = \Omega_f \cup \Omega_p$ is isotropic and radially symmetric around the z-axis, located at the centre of the borehole.

To describe the propagation of waves in Ω_p we have chosen Biot's equations of motion, while in Ω_f we have used the standard equation of motion for compressible, inhomogeneous, inviscid fluids. Appropriate absorbing boundary conditions for the artificial boundaries of Ω_p are derived, making them transparent to waves arriving normally. The same type of absorbing boundary conditions are used for the artificial boundaries of Ω_f. For the surface contact between Ω_f and Ω_p we have chosen the boundary condition suggested in [14], which represents a way of including the effects of the mud cake in the wave field. The special cases of an open or closed interface are also considered in the model.

This paper is related to several previous works on the subject. The theory of propagation of waves in fluid-saturated porous media was formulated by Biot in several classic papers [1, 2]. Results on the existence, uniqueness and finite element approximation of the solution of Biot's equations were given in [15, 17]. Also, mixed finite elements for radially symmetric three-dimensional problems were presented and analysed in [12]. Finite element methods for wave propagation in systems composed of elastic solids with imbedded fluid-saturated porous media were given and analysed in [11, 16], and some numerical results using those algorithms were presented in [4].

Synthetic full waveform acoustic logs have already been obtained using different techniques. In [14] the problem was treated assuming that the system Ω is homogeneous in depth and the solution was obtained via Fourier-transform techniques and numerical integration. The same approach was used in [3] but with Ω_p being an elastic solid. In [18], the system Ω was allowed to be inhomogeneous in depth with Ω_p being again an elastic solid; an approximate solution was computed using finite-difference techniques.

The organisation of the paper is as follows. In §2 we present the model by giving the partial differential equations and the initial and boundary conditions. In §3 we derive the weak form of the model and then present the results on the existence and uniqueness of the solution of the differential problem. In §4 we describe the finite element spaces used for the spatial discretisation and then formulate an explicit finite element procedure by using a mass-lumping quadrature rule for the first- and second-order time-derivative terms in the weak formulation. Results on the stability and convergence of the method are also given. Finally in §5 we derive the absorbing boundary conditions used for the artificial boundaries of Ω_p.

2. The model

We shall consider the propagation of waves in a fluid-filled borehole Ω_f surrounded by a fluid-saturated porous medium Ω_p. For simplicity the whole system $\Omega = \Omega_f \cup \Omega_p$ will be assumed to be isotropic and radially symmetric around the z-axis, located at the centre of the borehole. The system is naturally described using cylindrical coordinates (r, θ, z). Without loss of generality, the artificial top and bottom boundaries of Ω_f and Ω_p can be chosen to be the sets

$$\Gamma_1 = \{(r, \theta, z) \in \partial\Omega_f : 0 \leq r \leq R_f^T, 0 \leq \theta < 2\pi, z = 0, \text{ or } 0 \leq r \leq R_f^B, 0 \leq \theta < 2\pi, z = z_B\},$$

and

$$\Gamma_{21} = \{(r, \theta, z) \in \partial\Omega_p : R_f^T \leq r \leq R_p, 0 \leq \theta < 2\pi, z = 0 \text{ or } R_f^B \leq r \leq R_p, 0 \leq \theta < 2\pi, z = z_B\}.$$

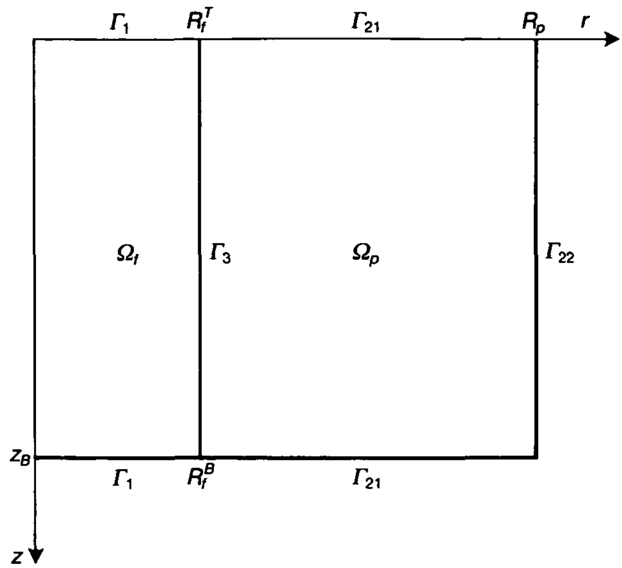

FIG. 1

Also, the artificial exterior boundary of Ω_p (and Ω) can be taken to be

$$\Gamma_{22} = \{(r, \theta, z) \in \partial\Omega_p : r = R_p,\ 0 \leqslant \theta < 2\pi,\ 0 \leqslant z \leqslant z_B\}.$$

Let Γ_3 denote the surface contact between Ω_f and Ω_p, which may have arbitrary shape in order to allow variations in the diameter of the borehole along the z-direction. A vertical cross-section of Ω for any fixed $\theta = \theta_0$ is shown in Fig. 1.

Assume cylindrical symmetry, and let $u_1 = (u_{1r}, 0, u_{1z})$ be the fluid displacement in Ω_f, let $u_2 = (u_{2r}, 0, u_{2z})$ be the solid displacement in Ω_p, and let $\tilde{u}_3 = (\tilde{u}_{3r}, 0, \tilde{u}_{3z})$ be the (averaged) fluid displacement in Ω_p. Let

$$u_3 = \phi(x)(\tilde{u}_3 - u_2) = (u_{3r}, 0, u_{3z}),$$

where $\phi(x)$ is the effective porosity, and set $u = (u_1, u_2, u_3)$. Here u_{ki} represents the displacement in the i-direction for $k = 1, 2, 3$. Next, because of the cylindrical symmetry assumption, the physical components of the strain tensor $\varepsilon(u_2)$ in the solid part of Ω_p are given [8, p. 114] by

$$\varepsilon_{rr}(u_2) = \frac{\partial u_{2r}}{\partial r}, \qquad \varepsilon_{\theta\theta}(u_2) = \frac{u_{2r}}{r}, \qquad \varepsilon_{zz}(u_2) = \frac{\partial u_{2z}}{\partial z},$$

$$\varepsilon_{rz}(u_2) = \tfrac{1}{2}\left(\frac{\partial u_{2r}}{\partial z} + \frac{\partial u_{2z}}{\partial r}\right),$$

$$\varepsilon_{r\theta}(u_2) = \tfrac{1}{2}\left(\frac{1}{r}\frac{\partial u_{2r}}{\partial \theta} + \frac{\partial u_{2\theta}}{\partial r} - \frac{u_{2\theta}}{r}\right) = 0,$$

$$\varepsilon_{\theta z}(u_2) = \tfrac{1}{2}\left(\frac{\partial u_{2\theta}}{\partial z} + \frac{1}{r}\frac{\partial u_{2z}}{\partial \theta}\right) = 0.$$

Also, note that

$$\nabla \cdot u_2 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = \frac{1}{r}\frac{\partial(ru_{2r})}{\partial r} + \frac{\partial u_{2z}}{\partial z}.$$

Let $\tau(u_2, u_3)$ and $p(u_2, u_3)$ denote the total stress tensor and the fluid pressure in Ω_p, respectively. Then the stress–strain relations in Ω_p can be written as follows [2]:

$$\left.\begin{aligned} &\tau_{rr}(u_2, u_3) = A\nabla \cdot u_2 + 2N\varepsilon_{rr}(u_2) + Q\nabla \cdot u_3, \\ &\tau_{\theta\theta}(u_2, u_3) = A\nabla \cdot u_2 + 2N\varepsilon_{\theta\theta}(u_2) + Q\nabla \cdot u_3, \\ &\tau_{zz}(u_2, u_3) = A\nabla \cdot u_2 + 2N\varepsilon_{zz}(u_2) + Q\nabla \cdot u_3, \\ &\tau_{rz}(u_2) = 2N\varepsilon_{rz}(u_2), \\ &\tau_{r\theta} = \tau_{\theta z} = 0, \\ &p(u_2, u_3) = -Q\nabla \cdot u_2 - H\nabla \cdot u_3. \end{aligned}\right\} \tag{2.1}$$

In these expressions, the coefficients A, N, Q and H are assumed to be functions of r and z alone.

Next, the strain-energy density $W_p(u_2, u_3)$ in Ω_p is given [2] by

$$W_p(u_2, u_3) = \tfrac{1}{2}[\tau_{rr}\varepsilon_{rr} + \tau_{\theta\theta}\varepsilon_{\theta\theta} + \tau_{zz}\varepsilon_{zz} + 2\tau_{rz}\varepsilon_{rz} - p\nabla \cdot u_3]. \tag{2.2}$$

Since W_p has to be a quadratic, positive-definite form in ε_{rr}, $\varepsilon_{\theta\theta}$, ε_{zz}, ε_{rz} and $\nabla \cdot u_3$, it is easily seen that the coefficients A, N, Q and H must satisfy the conditions

$$\left.\begin{aligned} &N(r, z) > 0, \quad H(r, z) > 0, \quad (A + \tfrac{2}{3}N)(r, z) > 0, \\ &(A + \tfrac{2}{3}N - Q^2/H)(r, z) > 0, \quad (r, \theta, z) \in \bar{\Omega}_p = \Omega_p \cup \partial\Omega_p. \end{aligned}\right\} \tag{2.3}$$

Also, dW_p must be an exact differential, so that

$$\left.\begin{aligned} &\tau_{rr} = \frac{\partial W_p}{\partial \varepsilon_{rr}}, \quad \tau_{\theta\theta} = \frac{\partial W_p}{\partial \varepsilon_{\theta\theta}}, \quad \tau_{zz} = \frac{\partial W_p}{\partial \varepsilon_{zz}}, \\ &\tau_{rz} = \frac{\partial W_p}{\partial \varepsilon_{rz}}, \quad p = \frac{\partial W_p}{\partial(-\nabla \cdot u_3)}. \end{aligned}\right\} \tag{2.4}$$

Let $E_p(r, z) \in \mathbb{R}^{5\times 5}$ be the symmetric, positive-definite matrix associated with W_p and set

$$Y(u_2, u_3) = (\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}, \nabla \cdot u_3, \varepsilon_{rz})^{\mathrm{T}},$$

so that

$$W_p(r, z) = \tfrac{1}{2}[E_pY, Y]_e,$$

where $[\ ,\ ]_e$ denotes the usual scalar product in $\mathbb{R}^n$.

For any matrix $D(r, z) \in \mathbb{R}^{n\times n}$, let $\lambda_{\min}(D(r, z))$ and $\lambda_{\max}(D(r, z))$ denote the minimum and maximum eigenvalues of $D(r, z)$ and set

$$\lambda_{\min}(D) = \inf_{r,z} \lambda_{\min}(D(r, z)), \qquad \lambda_{\max}(D) = \sup_{r,z} \lambda_{\max}(D(r, z)).$$

Under the conditions (2.3),

$$0<\lambda_{\min}(E_p)\leqslant\lambda_{\max}(E_p)<\infty,$$

and consequently,

$$W_p(r,z)\geqslant\frac{\lambda_{\min}(E_p)}{2}((\varepsilon_{rr})^2+(\varepsilon_{\theta\theta})^2+(\varepsilon_{zz})^2+(\varepsilon_{rz})^2+(\nabla\cdot u_3)^2)$$
$$\geqslant\frac{\lambda_{\min}(E_p)}{4}((\varepsilon_{rr})^2+(\varepsilon_{\theta\theta})^2+(\varepsilon_{zz})^2+2(\varepsilon_{rz})^2+(\nabla\cdot u_3)^2). \quad (2.5)$$

Next, let $\rho=\rho(r,z)$ denote the total mass density of bulk material in Ω_p and let $\rho_f=\rho_f(r,z)$ be the mass density of fluid both in Ω_f and Ω_p. Also, let $g=g(r,z)$ be a mass-coupling parameter between fluid and solid in Ω_p [2]. Assume that

$$\rho g(r,z)-\rho_f^2(r,z)>0, \qquad (r,\theta,z)\in\bar{\Omega}_p, \quad (2.6)$$

which is a necessary and sufficient condition for the kinetic-energy density in Ω_p to be positive.

Let $\mu=\mu(r,z)$ denote the fluid viscosity and let $k=k(r,z)$ denote the (scalar) rock permeability in Ω_p. Both μ and k will be assumed to be bounded above and below by positive constants.

Finally, let $A_f=A_f(r,z)$ denote the incompressibility modulus of the fluid in Ω_f, assumed to be bounded above and below by positive constants:

$$0<A_{f^*}\leqslant A_f(r,z)\leqslant A_f^*<\infty.$$

Then, we consider the following problem. Let

$$u_1^0(r,z)=(u_{1r}^0,0,u_{1z}^0),\quad v_1^0=(v_{1r}^0,0,v_{1z}^0),\quad f_1(r,z,t)=(f_{1r},0,f_{1z})$$

be given for $(r,\theta,z)\in\Omega_f$ and let

$$u_2^0(r,z)=(u_{2r}^0,0,u_{2z}^0),\quad u_3^0(r,z)=(u_{3r}^0,0,u_{3z}^0),$$
$$v_2^0(r,z)=(v_{2r}^0,0,v_{2z}^0),\quad v_3^0(r,z)=(v_{3r}^0,0,v_{3z}^0),$$

and

$$f_2(r,z)=(f_{2r},0,f_{2z}),\quad f_3(r,z)=(f_{3r},0,f_{3z})$$

be given for $(r,\theta,z)\in\Omega_p$. Then we want to find $u(r,z,t)=(u_1,u_2,u_3)$, $t\in J=(0,T)$, such that

$$\left.\begin{aligned}&\text{(i)} && \rho_f\frac{\partial^2u_{1r}}{\partial t^2}-\frac{\partial}{\partial r}(A_f\nabla\cdot u_1)=f_{1r}(r,z,t),\\ &\text{(ii)} && \rho_f\frac{\partial^2u_{1z}}{\partial t^2}-\frac{\partial}{\partial z}(A_f\nabla\cdot u_1)=f_{1z}(r,z,t)\end{aligned}\right\} \quad (2.7a)$$

for $(r, \theta, z, t) \in \Omega_f \times J$, and

$$\left.\begin{array}{ll}
\text{(iii)} & \rho\dfrac{\partial^2 u_{2r}}{\partial t^2}+\rho_f\dfrac{\partial^2 u_{3r}}{\partial t^2}-\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau_{rr}(u_2, u_3)) \\
& \qquad -\dfrac{\partial \tau_{rz}(u_2)}{\partial z}+\dfrac{\tau_{\theta\theta}(u_2, u_3)}{r}=f_{2r}(r, z, t), \\
\text{(iv)} & \rho\dfrac{\partial^2 u_{2z}}{\partial t^2}+\rho_f\dfrac{\partial^2 u_{3z}}{\partial t^2}-\dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau_{rz}(u_2))-\dfrac{\partial}{\partial z}\tau_{zz}(u_2, u_3)=f_{2z}(r, z, t), \\
\text{(v)} & \rho_f\dfrac{\partial^2 u_{2r}}{\partial t^2}+g\dfrac{\partial^2 u_{3r}}{\partial t^2}+\dfrac{\mu}{k}\dfrac{\partial u_{3r}}{\partial t}+\dfrac{\partial}{\partial r}p(u_2, u_3)=f_{3r}(r, z, t), \\
\text{(vi)} & \rho_f\dfrac{\partial^2 u_{2z}}{\partial t^2}+g\dfrac{\partial^2 u_{3z}}{\partial t^2}+\dfrac{\mu}{k}\dfrac{\partial u_{3z}}{\partial t}+\dfrac{\partial}{\partial z}p(u_2, u_3)=f_{3z}(r, z, t)
\end{array}\right\} \quad (2.7\text{b})$$

for $(r, \theta, z, t) \in \Omega_p \times J$, with boundary conditions

$$\left.\begin{array}{ll}
\text{(i)} & -A_f\nabla\cdot u_1=(\rho_f A_f)^{\frac{1}{2}}\dfrac{\partial u_1}{\partial t}\cdot \nu_f,\ (r, \theta, z, t)\in\Gamma_1\times J, \\
\text{(ii)} & (-\tau\nu_p\cdot\nu_p, -\tau\nu_p\cdot\chi_p^1, -\tau\nu_p\cdot\chi_p^2, p)^{\mathrm{T}} \\
& \quad =B\left(\dfrac{\partial u_2}{\partial t}\cdot\nu_p, \dfrac{\partial u_2}{\partial t}\cdot\chi_p^1, \dfrac{\partial u_2}{\partial t}\cdot\chi_p^2, \dfrac{\partial u_3}{\partial t}\cdot\nu_p\right)^{\mathrm{T}}, \\
& (r, \theta, z, t)\in(\Gamma_{21}\cup\Gamma_{22})\times J=\Gamma_2\times J, \\
\text{(iii)} & \tau\nu_p+A_f\nabla\cdot u_1\nu_f=0,\ (r, \theta, z, t)\in\Gamma_3\times J, \\
\text{(iv)} & (u_2+u_3)\cdot\nu_p+u_1\cdot\nu_f=0,\ (r, \theta, z, t)\in\Gamma_3\times J, \\
\text{(v)} & -p+m\dfrac{\partial u_3}{\partial t}\cdot\nu_p=A_f\nabla\cdot u_1,\ (r, \theta, z, t)\in\Gamma_3\times J,
\end{array}\right\} \quad (2.8)$$

and initial conditions

$$\left.\begin{array}{ll}
\text{(i)} & u_1(r, z, 0)=u_1^0(r, z),\ (r, \theta, z)\in\Omega_f, \\
\text{(ii)} & (u_2, u_3)(r, z, 0)=(u_2^0, u_3^0)(r, z),\ (r, \theta, z)\in\Omega_p, \\
\text{(iii)} & \dfrac{\partial u_1}{\partial t}(r, z, 0)=v_1^0(r, z),\ (r, \theta, z)\in\Omega_f, \\
\text{(iv)} & \dfrac{\partial(u_2, u_3)}{\partial t}(r, z, 0)=(v_2^0, v_3^0)(r, z),\ (r, \theta, z)\in\Omega_p.
\end{array}\right\} \quad (2.9)$$

In the above, $\nu_i=(\nu_{ir}, \nu_{i\theta}, \nu_{iz})=(\nu_{ir}, 0, \nu_{iz})$, $i=f, \rho$, denotes the unit outward normal along $\partial\Omega_i$, and χ_p^m, $m=1, 2$, denotes orthogonal unit tangent vectors along $\partial\Omega_p$. Also, $\tau\nu_p$ denotes the stress tensor on $\partial\Omega_p$ and $\tau\nu_p\cdot\nu_p$ and $\tau\nu_p\cdot\chi_p^m$, $m=1, 2$, are the normal and two tangent components of $\tau\nu_p$ on $\partial\Omega_p$.

Equations (2.7a) are the standard equations of motion for compressible, inviscid, inhomogeneous fluids, while equations (2.7b) are Biot's equations of motion for the fluid-saturated porous medium Ω_p [1, 2]. The boundary condition (2.8.i) is simply the equation of momentum for Γ_1, so that waves arriving normally to Γ_1 will be absorbed completely (that is, passed through transparently). Equation (2.8.ii) is an absorbing boundary condition for the artificial boundary Γ_2 of Ω_p; this relation is derived in §5. Again, its effect is to absorb the energy of waves arriving normally to Γ_2. The matrix $B(r, z) \in \mathbb{R}^{4\times 4}$ in the right-hand side of (2.8.ii) is symmetric and positive definite. Equation (2.8.iii) states the continuity of the normal stress and the vanishing of tangential stresses along Γ_3, while (2.8.iv) expresses the continuity of the normal displacement on Γ_3.

Finally, (2.8.v) relates the fluid pressure on both sides of Γ_3. This boundary condition is suggested in [14] to describe the behaviour of the mud cake using the non-negative coefficient $m = m(z)$ representing a surface impedance. The analysis of the model will be carried out for the case in which $0 < m_* \leq m(z) \leq m^* < \infty$, and we shall indicate briefly the change in the argument for the limit cases $m = 0$ and $m = +\infty$ corresponding to an open or sealed interface, respectively. Note that in the case of an open interface ($m = 0$), (2.8.v) simply states the continuity of the fluid pressure on Γ_3. Such a boundary condition was analysed in [10] and was shown to be energy-flux preserving. For a sealed interface ($m = +\infty$), it is necessary that

$$u_3 \cdot \nu_p = 0, \qquad (r, \theta, z, t) \in \Gamma_3 \times J. \tag{2.10}$$

In this case (2.8.v) should be replaced by (2.10), and (2.8.iv) reduces to

$$u_2 \cdot \nu_p + u_1 \cdot \nu_f = 0, \qquad (r, \theta, z, t) \in \Gamma_3 \times J.$$

3. The existence and uniqueness results

For $\Omega_i = \Omega_f$ or Ω_p let

$$(\varphi, \psi)_i = \int_{\Omega_i} \varphi(r, \theta, z)\psi(r, \theta, z) r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z$$

and

$$\|\varphi\|_{0,\Omega_i} = [(\varphi, \varphi)_i]^{\frac{1}{2}}$$

denote the inner product and norm in $L^2(\Omega_i)$. For any $\Gamma \subset \partial\Omega_i$ let

$$\langle v, w \rangle_\Gamma = \int_\Gamma v w \, \mathrm{d}\sigma$$

denote the inner product in $L^2(\Gamma)$, where $\mathrm{d}\sigma$ is the surface measure on Γ. Also, if $\varphi = (\varphi_r, \varphi_\theta, \varphi_z)$ and $\psi = (\psi_r, \psi_\theta, \psi_z)$, we shall denote by

$$(\varphi, \psi)_i = (\varphi_r, \psi_r)_i + (\varphi_\theta, \psi_\theta)_i + (\varphi_z, \psi_z)_i$$

and

$$\|\varphi\|_{0,\Omega_i} = [(\varphi, \varphi)_i]^{\frac{1}{2}}$$

the inner product and norm in $L^2(\Omega_i)^3$.

Next, let

$$H(\mathrm{div}, \Omega_i) = \{\varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in L^2(\Omega_i)^3 : \nabla \cdot \varphi \in L^2(\Omega_i)\},$$

provided with the natural norm

$$\|\varphi\|_{H(\mathrm{div},\Omega_i)} = [\|\varphi\|_{0,\Omega_i}^2 + \|\nabla\cdot\varphi\|_{0,\Omega_i}^2]^{\frac{1}{2}}.$$

Set

$$\tilde{H}(\mathrm{div},\Omega_i) = \{\varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in H(\mathrm{div},\Omega_i) \colon \varphi_\theta = 0\},$$

which is a closed subspace of $H(\mathrm{div},\Omega_i)$. Note that $\varphi \in H(\mathrm{div},\Omega_f)$ implies that $\varphi_r|_{r=0} = 0$. Also, set

$$\begin{aligned}\tilde{H}^1(\Omega_p)^3 &= \{\varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in H^1(\Omega_p)^3 \colon \varphi_\theta = 0,\ \partial\varphi_r/\partial\theta = \partial\varphi_z/\partial\theta = 0\}\\ &= \{\varphi = (\varphi_r, \varphi_\theta, \varphi_z) \in H^1(\Omega_p)^3 \colon \varphi_\theta = 0,\ \varepsilon_{r\theta}(\varphi) = \varepsilon_{\theta z}(\varphi) = 0\}.\end{aligned}$$

Note that, for any $\varphi \in \tilde{H}^1(\Omega_p)^3$, standard calculus shows that in $\mathbb{R}^3$ with cylindrical symmetry,

$$\begin{aligned}\|\varphi\|_{1,\Omega_p} = \Bigg[\int_{\Omega_p} \Bigg[(\varphi_r)^2 + (\varphi_z)^2 + \left(\frac{\partial\varphi_r}{\partial r}\right)^2 + \left(\frac{\partial\varphi_z}{\partial z}\right)^2 + \left(\frac{\varphi_r}{r}\right)^2\\ + \left(\frac{\partial\varphi_z}{\partial r}\right)^2 + \left(\frac{\partial\varphi_r}{\partial r}\right)^2\Bigg] r\,dr\,d\theta\,dz\Bigg]^{\frac{1}{2}}.\end{aligned}$$

It is clear that $\tilde{H}^1(\Omega_p)^3$ is a closed subspace of $H^1(\Omega_p)^3$.

Next, let $\tilde{V} = \tilde{H}(\mathrm{div},\Omega_f) \times \tilde{H}^1(\Omega_p)^3 \times \tilde{H}(\mathrm{div},\Omega_p)$, which is a separable Hilbert space under the norm

$$\|v\|_{\tilde{V}} = [\|v_1\|_{\tilde{H}(\mathrm{div},\Omega_f)}^2 + \|v_2\|_{1,\Omega_p}^2 + \|v_3\|_{\tilde{H}(\mathrm{div},\Omega_p)}^2]^{\frac{1}{2}}.$$

Since the boundary condition (2.8.iv) will be imposed strongly, we shall restrict the admissible test functions to the set

$$V = \{v = (v_1, v_2, v_3) \in \tilde{V} \colon (v_2 + v_3 - v_1)\cdot\nu_f = 0 \text{ on } \Gamma_3\};$$

V is a closed, separable subspace of $\tilde{V}$ (with the same norm).

The weak form of problem (2.7), (2.8), and (2.9) is obtained as usual by testing equations (2.7) against any admissible function $v = (v_1, v_2, v_3) \in V$, using integration by parts and applying the boundary conditions (2.8.i), (2.8.ii), (2.8.iii), and (2.8.v). In doing so, we obtain

$$\begin{aligned}&\left(\rho_f \frac{\partial^2 u_1}{\partial t^2}, v_1\right)_f + \left(\mathcal{A}\frac{\partial^2(u_2,u_3)}{\partial t^2}, (v_2,v_3)\right)_p + \left(\mathcal{C}\frac{\partial(u_2,u_3)}{\partial t}, (v_2,v_3)\right)_p + \Lambda(u,v)\\ &\quad + \left\langle(\rho_f A_f)^{\frac{1}{2}}\frac{\partial u_1}{\partial t}\cdot\nu_f, v_1\cdot\nu_f\right\rangle_{\Gamma_1} + \Bigg\langle B\left(\frac{\partial u_2}{\partial t}\cdot\nu_p, \frac{\partial u_2}{\partial t}\cdot\chi_p^1, \frac{\partial u_2}{\partial t}\cdot\chi_p^2, \frac{\partial u_3}{\partial t}\cdot\nu_p\right)^{\mathrm{T}},\\ &\qquad (v_2\cdot\nu_p, v_2\cdot\chi_p^1, v_2\cdot\chi_p^2, v_3\cdot\nu_p)\Bigg\rangle_{\Gamma_2} + \left\langle m\frac{\partial u_3}{\partial t}\cdot\nu_p, v_3\cdot\nu_p\right\rangle_{\Gamma_3}\\ &= (f_1, v_1)_f + ((f_2, f_3), (v_2, v_3))_p, \qquad v = (v_1, v_2, v_3) \in V,\ t \in J. \qquad (3.1)\end{aligned}$$

Here $\mathcal{A}(r,z)$ and $\mathcal{C}(r,z)$ are matrices in $\mathbb{R}^{4\times 4}$ given by

$$\mathcal{A} = \begin{pmatrix}\rho I & \rho_f I\\ \rho_f I & g I\end{pmatrix}, \qquad \mathcal{C} = \mu k^{-1}\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix},$$

I being the identity matrix in $\mathbb{R}^{2\times 2}$. Note that $\mathscr{C}$ is non-negative and $\mathscr{A}$ is positive definite, thanks to (2.6). Also, $\Lambda(v, w)$ is the symmetric, bilinear form defined on $\tilde{V}$ by

$$\begin{aligned}\Lambda(v, w) = {} & (A_f \nabla \cdot v_1, \nabla \cdot w_1)_f + (\tau_{rr}(v_2, v_3), \varepsilon_{rr}(w_2))_p \\ & + (\tau_{\theta\theta}(v_2, v_3), \varepsilon_{\theta\theta}(w_2))_p + (\tau_{zz}(v_2, v_3), \varepsilon_{zz}(w_2))_p \\ & + 2(\tau_{rz}(v_2, v_3), \varepsilon_{rz}(w_2))_p \\ & - (p(v_2, v_3), \nabla \cdot w_3)_p \qquad \text{for } v, w \in \tilde{V}.\end{aligned}$$

Note that combining (2.2), (2.5), and Korn's second inequality [6, 7, 13] implies that

$$\begin{aligned}\Lambda(v, v) \geq {} & A_{f^*} \|\nabla \cdot v_1\|^2_{0,\Omega_f} + \frac{\lambda_{\min}(E_p)}{2} \int_{\Omega_p} [(\varepsilon_{rr}(v_2))^2 + (\varepsilon_{\theta\theta}(v_2))^2 \\ & + (\varepsilon_{zz}(v_2))^2 + 2(\varepsilon_{rz}(v_2))^2 + (\nabla \cdot v_3)^2] r \, dr \, d\theta \, dz \\ \geq {} & c_1 \|v\|^2_{\tilde{V}} - c_2(\|v_1\|^2_{0,\Omega_f} + \|(v_2, v_3)\|^2_{0,\Omega_p}), \qquad v \in \tilde{V}. \end{aligned} \tag{3.2}$$

Let $\gamma \geq c_2$ be any fixed constant and let Λ_γ be the bilinear symmetric form defined over $\tilde{V}$ by

$$\Lambda_\gamma(v, w) = \Lambda(v, w) + \gamma[(v_1, w_1)_f + ((v_2, v_3), (w_2, w_3))_p].$$

Then Λ_γ is $\tilde{V}$-continuous and $\tilde{V}$-coercive.

Next, set

$$Q_s^2 = \left\| \frac{\partial^s f_1}{\partial t^s} \right\|^2_{L^2(J;(L^2(\Omega_f))^2)} + \left\| \frac{\partial^s (f_2, f_3)}{\partial t^s} \right\|^2_{L^2(J;(L^2(\Omega_p))^4)},$$

$$G_0^2 = \|u_1^0\|^2_{2,\Omega_f} + \|(u_2^0, u_3^0)\|^2_{2,\Omega_p} + \|v^0\|^2_{\tilde{V}} + \|f_1(0)\|^2_{0,\Omega_f} + \|(f_2(0), f_3(0))\|^2_{0,\Omega_p} + 1.$$

The well-posedness of problem (2.7), (2.8), and (2.9) follows from the following theorem.

THEOREM 3.1 *Let $f = (f_1, f_2, f_3)$, $u^0 = (u_0^1, u_0^2, u_3^0)$ and $v^0 = (v_1^0, v_2^0, v_3^0)$ be given and such that $G_0 < \infty$, $Q_i < \infty$, $i = 0, 1$. Assume that Γ_3 is of class C^m for some integer $m \geq 2$. Also, assume that*

$$\text{support}(u_1^0) \cap \Omega_f \Subset \Omega_f, \qquad \text{support}(v_1^0) \cap \Omega_f \Subset \Omega_f,$$
$$\text{support}(u_2^0, u_3^0) \Subset \Omega_p, \qquad \text{support}(v_2^0, v_3^0) \Subset \Omega_p.$$

Then there exists a unique solution $u(r, z, t)$ of problem (2.7), (2.8), and (2.9) such that u, $\partial u/\partial t \in L^\infty(J, V)$; $\partial^2 u_1/\partial t^2 \in L^2(J, L^2(\Omega_f)^2)$; and $\partial^2(u_2, u_3)/\partial t^2 \in L^2(J, L^2(\Omega_p)^4)$.

Proof. Let

$$\tilde{H}^2(\Omega_f)^3 = \{\varphi \in H^2(\Omega_f)^3 \colon \varphi_r|_{r=0} = 0,\ \varphi_\theta = 0,\ \partial\varphi_r/\partial\theta = \partial\varphi_z/\partial\theta = 0\},$$
$$\tilde{H}^2(\Omega_p)^3 = \{\varphi \in H^2(\Omega_p)^3 \colon \varphi_\theta = 0,\ \partial\varphi_r/\partial\theta = \partial\varphi_z/\partial\theta = 0\},$$

and set

$$E = \tilde{H}^2(\Omega_f)^3 \times \tilde{H}^2(\Omega_p)^3 \times \tilde{H}^2(\Omega_p)^3.$$

Clearly, $E \subset \tilde{V}$ and the argument given in [16] can be used here to show that $E \cap V$ is dense in V. The compactness argument given in [15, 16] can be used with minor modifications to obtain the conclusions of the theorem.

In the case in which the contact surface Γ_3 between Ω_f and Ω_p is known just to be Lipchitz continuous, the following existence and uniqueness theorem holds, its proof being similar to that of Theorem 3.1.

THEOREM 3.2 *Let* $f = (f_1, f_2, f_3)$ *be given and such that* $Q_i < \infty$, $i = 0, 1$. *Assume that* $u^0 = v^0 = 0$ *and that* Γ_3 *is Lipchitz continuous. Then there exists a unique solution* $u(r, z, t)$ *of problem* (2.7), (2.8), *and* (2.9) *such that* u, $\partial u/\partial t \in L^\infty(J, V)$; $\partial^2 u_1/\partial t^2 \in L^\infty(J, L^2(\Omega_f)^2)$; *and* $\partial^2(u_2, u_3)/\partial t^2 \in L^\infty(J, L^2(\Omega_p)^4)$.

Finally, let us indicate the modifications needed to treat the cases of an open or a sealed interface Γ_3. For the open interface ($m = 0$) the original space V is adequate. For the sealed interface ($m = +\infty$) the space V should be chosen to be

$$V = \{v = (v_1, v_2, v_3) \in \bar{V} : (v_2 - v_1) \cdot \nu_f = 0,\ v_3 \cdot \nu_f = 0 \text{ on } \Gamma_3\}.$$

Thus, in both cases the weak form (3.1) remains formally unchanged, except that the last term in the left-hand side disappears. Also, the conclusions of Theorems 3.1 and 3.2 remain valid.

4. An explicit finite element procedure

For $0 < h < 1$, let $\tau_h^f = \tau_h^f(\Omega_f)$ and $\tau_h^p = \tau_h^p(\Omega_p)$ be quasiregular partitions of Ω_f and Ω_p into elements generated by the rotation around the z-axis of rectangles in the (r, z)-variables of diameter bounded by h. Set $\tau_h = \tau_h^f \cup \tau_h^p$. Since the boundary condition (2.8.iv) will be imposed strongly on the finite element spaces to be used for the spatial discretisation, the partitions τ_h^f and τ_h^p will be assumed to be compatible along the contact surface Γ_3 in the following sense. For any vertical cross-section $\tau_h \cap \{\theta = \theta_0\}$ of τ_h, if R_f is a rectangle in $\tau_h^f \cap \{\theta = \theta_0\}$ such that one edge e of R_f is contained in Γ_3, then e is also an edge of some rectangle R_p in $\tau_h^p \cap \{\theta = \theta_0\}$. Let $P_{1,1}(r, z)$ denote the bilinear polynomials in the (r, z)-variables and set

$$\mathcal{M}_h = \{\varphi = (\varphi_r, 0, \varphi_z) \in C^0(\bar{\Omega}_p) : \varphi_r \in rP_{1,1}(r, z) \text{ and } \varphi_z \in P_{1,1}(r, z)\}.$$

Then, $\mathcal{M}_h \subset \tilde{H}^1(\Omega_p)^3$.

The r-component of φ is multiplied by r in order to ensure that all components of the strain tensor of φ remain polynomials in r and z. It does not affect the approximation property

$$\inf_{\varphi \in \mathcal{M}_h} [\|v - \varphi\|_{0,\Omega_p} + h\, \|v - \varphi\|_{1,\Omega_p}] \leq ch^s\, \|v\|_{s,\Omega_p}, \qquad s = 1, 2; \tag{4.1}$$

this result is proved in [12].

Let $W_h(\Omega_i)$, $i = f, p$, be the vector part of the lowest-order mixed finite element space associated with τ_h^i defined by Morley [12]. Away from $r = 0$, the elements in $W_h(\Omega_i)$ are locally of the form $(ar^{-1} + br, 0, c + dz)$, while the innermost elements near $r = 0$ have the local form $(br, 0, c + dz)$. Globally the elements must lie in $H(\text{div}, \Omega_j)$, $j = f$ or p, as appropriate. Note that the divergence of each element is piecewise constant. It is shown in [12] that

$$\left.\begin{aligned}&\text{(i)}\quad \inf_{\varphi\in W_h(\Omega_i)} \|v-\varphi\|_{H(\mathrm{div},\Omega_i)} \le ch(\|v\|_{1,\Omega_i} + \|\nabla\cdot v\|_{1,\Omega_i}),\\ &\text{(ii)}\quad \inf_{\varphi\in W_h(\Omega_i)} \|v-\varphi\|_{0,\Omega_i} \le ch\,\|v\|_{1,\Omega_i}.\end{aligned}\right\} \tag{4.2}$$

Let $\tilde{V}_h = W_h(\Omega_f)\times \mathcal{M}_h \times W_h(\Omega_p)$ and set $V_h = \{v\in \tilde{V}_h : (v_2+v_3-v_1)\cdot \nu_f = 0 \text{ on } \Gamma_3\}$. Then $V_h \subset V$ and it follows from (4.1) and (4.2) that

$$\inf_{\varphi\in V_h} [\|v_1-\varphi_1\|_{0,\Omega_f} + \|(v_2,v_3)-(\varphi_2,\varphi_3)\|_{0,\Omega_p}] \le ch[\|v_1\|_{1,\Omega_f} + \|(v_2,v_3)\|_{1,\Omega_p}] \tag{4.3}$$

for $v\in (\tilde{\mathrm{H}}^1(\Omega_f)^3 \times \tilde{\mathrm{H}}^1(\Omega_p)^3 \times \tilde{\mathrm{H}}^1(\Omega_p)^3)\cap V$ and that

$$\inf_{\varphi\in V_h} \|v-\varphi\|_V \le ch[\|v_1\|_{1,\Omega_f} + \|\nabla\cdot v_1\|_{1,\Omega_f} + \|v_2\|_{2,\Omega_p} + \|v_3\|_{1,\Omega_p} + \|\nabla\cdot v_3\|_{1,\Omega_p}] \tag{4.4}$$

for $v\in (\tilde{\mathrm{H}}^1(\Omega_f)^3 \times \tilde{\mathrm{H}}^2(\Omega_p)^3 \times \tilde{\mathrm{H}}^1(\Omega_p)^3)\cap V$ such that $\nabla\cdot v_1 \in \mathrm{H}^1(\Omega_f)$ and $\nabla\cdot v_3 \in \mathrm{H}^1(\Omega_p)$.

Let L be a positive integer, $\Delta t = T/L$, and $U^n = U(n\,\Delta t)$. Set

$$d_t U^n = (U^{n+1}-U^n)/\Delta t, \qquad \partial U^n = (U^{n+1}-U^{n-1})/2\,\Delta t,$$
$$\partial^2 U^n = (U^{n+1}-2U^n+U^{n-1})/(\Delta t)^2.$$

Since we want to use an explicit procedure, we shall compute all integrals involving time-derivative terms using the quadrature rule

$$\int_Q f(r,z)r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \approx \frac{2\pi}{4} h_r h_z [f_1 r_1 + f_2 r_2 + f_3 r_3 + f_4 r_4], \tag{4.5}$$

where f_i denotes the value of f at the node a_i in the rectangle Q (see Fig. 2). Note that the rule (4.5) is exact if $f(r,z)r$ is bilinear.

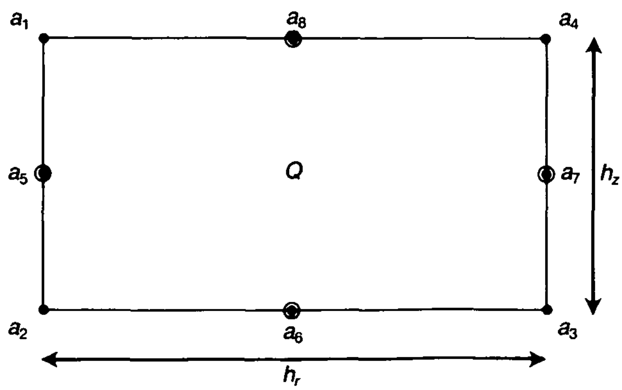

FIG. 2

For the elements in $\mathcal{M}_h$, the rule (4.5) is the natural choice since the local degrees of freedom for any element $v=(v_r, 0, v_z)$ in $\mathcal{M}_h$ are the values of v_r and v_z at the nodes a_i, $1\leq i\leq 4$. On the other hand, since the local degrees of freedom of a mixed Morley element $v=(v_r, 0, v_z)$ are the values of $v\cdot\nu_Q$ at the midpoints of each side of Q (that is, the values of v_r at the nodes a_5 and a_7 and of v_z at the nodes a_6 and a_8), such values being constant along the sides of Q, the mass-lumping quadrature rule (4.5) can be used for those elements as well.

Let $[v, w]_i$ and $|||v|||_{0,\Omega_i}$, $i=f, p$, denote the inner product $(v, w)_i$ and the norm $\|v\|_{0,\Omega_i}$ computed approximately using the quadrature rule (4.5). Also, let $\langle\langle v, w\rangle\rangle_\Gamma$ denote the inner product $\langle v, w\rangle_\Gamma$ computed using (4.5).

The discrete-time explicit Galerkin procedure is defined as follows. Find $U^n\in V_h$, $n=0,\ldots, L$, such that

$$\begin{aligned}
&[\rho_f\partial^2 U_1^n, v_1]_f+[\mathcal{A}\partial^2(U_2, U_3)^n, (v_2, v_3)]_p+[\mathcal{C}\partial(U_2, U_3)^n, (v_2, v_3)]_p\\
&\quad+\Lambda(U^n, v)+\langle\langle(\rho_f A_f)^{\frac12}\partial U_1^n\cdot\nu_f, v_1\cdot\nu_f\rangle\rangle_{\Gamma_1}\\
&\quad+\langle\langle B(\partial U_2^n\cdot\nu_p, \partial U_2^n\cdot\chi_p^1, \partial U_2^n\cdot\chi_p^2, \partial U_3^n\cdot\nu_p), (v_2\cdot\nu_p, v_2\cdot\chi_p^1, v_2\cdot\chi_p^2, v_3\cdot\nu_p)\rangle\rangle_{\Gamma_2}\\
&\quad+\langle\langle m\,\partial U_3^n\cdot\nu_p, v_3\cdot\nu_p\rangle\rangle_{\Gamma_3}\\
&\qquad\qquad=(f_1^n, v_1)_f+((f_2^n, f_3^n), (v_2, v_3))_p,\qquad v\in V_h,\ 1\leq n\leq L-1.
\end{aligned}\tag{4.6}$$

We shall analyse the stability of the scheme (4.6). The choice of the test function $v=\partial U^n$ in (4.6) gives us the inequality

$$\begin{aligned}
&\frac{1}{2\Delta t}\{|||\rho_f^{\frac12}d_tU_1^n|||_{0,\Omega_f}^2-|||\rho_f^{\frac12}d_tU_1^{n-1}|||_{0,\Omega_f}^2\\
&\quad+|||\mathcal{A}^{\frac12}\,d_t(U_2, U_3)^n|||_{0,\Omega_p}^2-|||\mathcal{A}^{\frac12}\,d_t(U_2, U_3)^{n-1}|||_{0,\Omega_p}^2\}\\
&\quad+[\mathcal{C}\partial(U_2, U_3)^n, \partial(U_2, U_3)^n]_p+\Lambda(U^n, \partial U^n)\\
&\quad+\langle\langle(\rho_f A_f)^{\frac12}\partial U_1^n\cdot\nu_f, \partial U_1^n\cdot\nu_f\rangle\rangle_{\Gamma_1}\\
&\quad+\langle\langle B(\partial U_2^n\cdot\nu_p, \partial U_2^n\cdot\chi_p^1, \partial U_2^n\cdot\chi_p^2, \partial U_3^n\cdot\nu_p),\\
&\qquad(\partial U_2^n\cdot\nu_p, \partial U_2^n\cdot\chi_p^1, \partial U_2^n\cdot\chi_p^2, \partial U_3^n\cdot\nu_p)\rangle\rangle_{\Gamma_2}\\
&\quad+\langle\langle m\,\partial U_3^n\cdot\nu_p, \partial U_3^n\cdot\nu_p\rangle\rangle_{\Gamma_3}\\
&\leq C\{\|f_1^n\|_{0,\Omega_f}^2+\|(f_2^n, f_3^n)\|_{0,\Omega_p}^2+\|d_tU_1^n\|_{0,\Omega_f}^2\\
&\quad+\|d_tU_1^{n-1}\|_{0,\Omega_f}^2+\|d_t(U_2, U_3)^n\|_{0,\Omega_p}^2+\|d_t(U_2, U_3)^{n-1}\|_{0,\Omega_p}^2\}.
\end{aligned}\tag{4.7}$$

Next, note that

$$\begin{aligned}
2\Delta t\Lambda(U^n, \partial U^n)=\tfrac12\{&\Lambda(U^{n+1}, U^{n+1})-\Lambda(U^{n-1}, U^{n-1})\\
&+\Lambda(U^n-U^{n-1}, U^n-U^{n-1})-\Lambda(U^{n+1}-U^n, U^{n+1}-U^n)\}.
\end{aligned}$$

Then, add

$$\begin{aligned}
&\frac{\gamma}{4\Delta t}\{\|U_1^{n+1}\|_{0,\Omega_f}^2-\|U_1^{n-1}\|_{0,\Omega_f}^2\}\\
&\leq\tfrac14\gamma\{\|d_tU_1^n\|_{0,\Omega_f}^2+\|d_tU_1^{n-1}\|_{0,\Omega_f}^2+\|U_1^{n+1}\|_{0,\Omega_f}^2+\|U_1^{n-1}\|_{0,\Omega_f}^2\}
\end{aligned}$$

and

$$\frac{\gamma}{4\Delta t}\{\|(U_2, U_3)^{n+1}\|^2_{0,\Omega_p} - \|(U_2, U_3)^{n-1}\|^2_{0,\Omega_p}\}$$
$$\leq \tfrac{1}{4}\gamma\{\|\mathrm{d}_t(U_2, U_3)^n\|^2_{0,\Omega_p} + \|\mathrm{d}_t(U_2, U_3)^{n-1}\|^2_{0,\Omega_p} + \|(U_2, U_3)^{n+1}\|^2_{0,\Omega_p} + \|(U_2, U_3)^{n-1}\|^2_{0,\Omega_p}\}$$

to (4.7), multiply by $2\Delta t$, and sum the resulting inequality from $n=1$ to $n=N$, $1\leq N\leq L-1$. Since $\mathscr{C}$ is a non-negative matrix and all the boundary terms in the left-hand side of (4.7) are non-negative,

$$\begin{aligned}
&|||\rho_f^{\frac{1}{2}}\mathrm{d}_t U_1^N|||^2_{0,\Omega_f} + |||\mathscr{A}^{\frac{1}{2}}\mathrm{d}_t(U_2, U_3)^N|||^2_{0,\Omega_p} - \tfrac{1}{2}(\Delta t)^2\Lambda(\mathrm{d}_t U^N, \mathrm{d}_t U^N)\\
&\quad + \tfrac{1}{2}\{\Lambda_\gamma(U^{N+1}, U^{N+1}) + \Lambda_\gamma(U^N, U^N)\}\\
&\leq \tfrac{1}{2}(\Delta t)^2\Lambda(\mathrm{d}_t U^0, \mathrm{d}_t U^0) + C\{|||\mathrm{d}_t U_1^0|||^2_{0,\Omega_f} + |||\mathrm{d}_t(U_2, U_3)^0|||^2_{0,\Omega_p} + \|U^0\|^2_V + \|U^1\|^2_V\\
&\quad + \sum_{n=1}^{N}[\|f_1^n\|^2_{0,\Omega_f} + \|(f_2, f_3)^n\|^2_{0,\Omega_p} + \|\mathrm{d}_t U_1^n\|^2_{0,\Omega_f}\\
&\quad + \|\mathrm{d}_t U_1^{n-1}\|^2_{0,\Omega_f} + \|\mathrm{d}_t(U_2, U_3)^n\|^2_{0,\Omega_p} + \|\mathrm{d}_t(U_2, U_3)^{n-1}\|^2_{0,\Omega_p}\\
&\quad + \|U_1^{n+1}\|^2_{0,\Omega_f} + \|U_1^{n-1}\|^2_{0,\Omega_f} + \|(U_2, U_3)^{n+1}\|^2_{0,\Omega_p}\\
&\quad + \|(U_2, U_3)^{n-1}\|^2_{0,\Omega_p}]\,\Delta t\}, \qquad 1\leq N\leq L-1.
\end{aligned} \tag{4.8}$$

Next, note that

$$\begin{aligned}
\Lambda(\mathrm{d}_t U^N, \mathrm{d}_t U^N) \leq A_f^*\,\|\nabla\cdot \mathrm{d}_t U_1^N\|^2_{0,\Omega_f} &+ \lambda_{\max}(E_p)\{\|\varepsilon_{rr}(\mathrm{d}_t U_2^N)\|^2_{0,\Omega_p}\\
&+ \|\varepsilon_{\theta\theta}(\mathrm{d}_t U_2^N)\|^2_{0,\Omega_p} + \|\varepsilon_{zz}(\mathrm{d}_t U_2^N)\|^2_{0,\Omega_p}\\
&+ \|\varepsilon_{rz}(\mathrm{d}_t U_2^N)\|^2_{0,\Omega_p} + \|\nabla\cdot \mathrm{d}_t U_3^N\|^2_{0,\Omega_p}\}.
\end{aligned} \tag{4.9}$$

Also note that there exists a constant c_3 independent of h such that the following inverse hypotheses hold:

$$\left.\begin{aligned}
&\text{(i)} \qquad \|\nabla\cdot v\|_{0,\Omega_i} \leq c_3 h^{-1}\,|||v|||_{0,\Omega_i},\ v\in W_h(\Omega_i),\ i=f, p,\\
&\text{(ii)} \qquad \{\|\varepsilon_{rr}(v)\|^2_{0,\Omega_p} + \|\varepsilon_{\theta\theta}(v)\|^2_{0,\Omega_p} + \|\varepsilon_{zz}(v)\|^2_{0,\Omega_p} + \|\varepsilon_{rz}(v)\|^2_{0,\Omega_p}\}^{\frac{1}{2}}\\
&\qquad\qquad \leq c_3 h^{-1}\,|||v|||_{0,\Omega_p},\ v\in \mathscr{M}_h.
\end{aligned}\right\} \tag{4.10}$$

For a uniform grid, a calculation shows that c_3 is not greater than 6·37; this may not be the best possible constant. In the general case c_3 will contain a factor that measures the quasiuniformity of the grid.

Then, since $\lambda_{\min}(\mathscr{A})>0$ (cf. (2.6)), it follows from (4.9) and (4.10) that

$$\begin{aligned}
&|||\rho_f^{\frac{1}{2}}\mathrm{d}_t U_1^N|||^2_{0,\Omega_f} + |||\mathscr{A}^{\frac{1}{2}}\mathrm{d}_t(U_2, U_3)^N|||^2_{0,\Omega_p} - \tfrac{1}{2}(\Delta t)^2\Lambda(\mathrm{d}_t U^N, \mathrm{d}_t U^N)\\
&\geq \left(\rho_{f^*} - \left(\frac{\Delta t}{h}\right)^2\frac{c_3^2}{2}A_f^*\right)|||\mathrm{d}_t U_1^N|||^2_{0,\Omega_f}\\
&\quad + \left(\lambda_{\min}(\mathscr{A}) - \left(\frac{\Delta t}{h}\right)^2\frac{c_3^2}{2}\lambda_{\max}(E_p)\right)|||\mathrm{d}_t(U_2, U_3)^N|||^2_{0,\Omega_p}\\
&\geq \tfrac{1}{2}\rho_{f^*}\,|||\mathrm{d}_t U_1^N|||^2_{0,\Omega_f} + \tfrac{1}{2}\lambda_{\min}(\mathscr{A})\,|||\mathrm{d}_t(U_2, U_3)^N|||^2_{0,\Omega_p},
\end{aligned} \tag{4.11}$$

where ρ_{f^*} is the minimum of $\rho_f(r, z)$ in Ω_f and provided that Δt and h satisfy the stability condition

$$\Delta t \leq \frac{h}{c_3} \min\left(\left(\frac{\rho_{f^*}}{A_f^*}\right)^{\frac{1}{2}}, \left(\frac{\lambda_{\min}(\mathscr{A})}{\lambda_{\max}(E_p)}\right)^{\frac{1}{2}}\right). \tag{4.12}$$

(A modification of the argument above would permit us to replace the term $(\lambda_{\min}(\mathscr{A})/\lambda_{\max}(E_p))^{\frac{1}{2}}$ by the reciprocal of the maximum wave velocity in Ω_p; the constant c_3 may be different in this case.)

Also, note that for Δt and h as in (4.12),

$$\tfrac{1}{2}(\Delta t)^2 \Lambda(\mathrm{d}_t U^0, \mathrm{d}_t U^0) \leq C[|||\mathrm{d}_t U_1^0|||_{0,\Omega_f}^2 + |||\mathrm{d}_t(U_2, U_3)^0|||_{0,\Omega_p}^2]. \tag{4.13}$$

Next, an easy calculation shows that there exists a constant c_4 independent of h such that

$$\|v_1\|_{0,\Omega_f} \leq c_4 |||v_1|||_{0,\Omega_f}$$

and

$$\|(v_2, v_3)\|_{0,\Omega_p} \leq c_4 |||(v_2, v_3)|||_{0,\Omega_p}$$

for any $v \in V_A$. Thus, using (4.11), (4.12), (4.13), and the V-coercivity of Λ_γ in (4.8), we see the following inequality holds:

$$\begin{aligned} &|||\mathrm{d}_t U_1^N|||_{0,\Omega_f}^2 + |||\mathrm{d}_t(U_2, U_3)^N|||_{0,\Omega_p}^2 + \|U^N\|_V^2 + \|U^{N+1}\|_V^2 \\ &\quad \leq C\{|||\mathrm{d}_t U_1^0|||_{0,\Omega_f}^2 + |||\mathrm{d}_t(U_2, U_3)^0|||_{0,\Omega_p}^2 + \|U^0\|_V^2 + \|U^1\|_V^2 \\ &\quad + \|f_1\|_{L^\infty(J, L^2(\Omega_f)^2)}^2 + \|(f_2, f_3)\|_{L^\infty(J, L^2(\Omega_p)^4)}^2 \\ &\quad + \sum_{n=1}^{N} [|||\mathrm{d}_t U_1^n|||_{0,\Omega_f}^2 + |||\mathrm{d}_t U_1^{n-1}|||_{0,\Omega_f}^2 + |||\mathrm{d}_t(U_2, U_3)^n|||_{0,\Omega_p}^2 \\ &\quad + |||\mathrm{d}_t(U_2, U_3)^{n-1}|||_{0,\Omega_p}^2 + \|U^{n+1}\|_V^2 + \|U^{n-1}\|_V^2]\,\Delta t\}, \qquad 1 \leq N \leq L-1. \end{aligned}$$

Then Gronwall's lemma implies that

$$\begin{aligned} \max_{1 \leq N \leq L-1} &(|||\mathrm{d}_t U_1^N|||_{0,\Omega_f} + |||\mathrm{d}_t(U_2, U_3)^N|||_{0,\Omega_p} + \|U^N\|_V) \\ &\leq C\{|||\mathrm{d}_t(U_1^0)|||_{0,\Omega_f} + |||\mathrm{d}_t(U_2 - U_3)^0|||_{0,\Omega_p} + \|U^0\|_V \\ &\quad + \|U^1\|_V + \|f_1\|_{L^\infty(J, L^2(\Omega_f)^2)} + \|(f_2, f_3)\|_{L^\infty(J, L^2(\Omega_p)^4)}^2\}, \end{aligned} \tag{4.14}$$

which shows that the scheme (4.6) is stable under the condition (4.12). It is also obvious that (4.14) gives us existence and uniqueness for the solution $(U^n)_{1 \leq n \leq L-1}$ of (4.6).

Finally, since the quadrature rule employed in the procedure is $O(h^2)$-correct, a standard argument combining the approximating properties (4.3) and (4.4) with the ideas leading to (4.14) would give us the optimal order error estimate

$$\begin{aligned} \max_{1 \leq N \leq L-1} &(|||\mathrm{d}_t(u_1 - U_1)^N|||_{0,\Omega_f} + |||\mathrm{d}_t(u_2 - U_2, u_3 - U_3)^N|||_{0,\Omega_p} + \|(u - U)^N\|_V) \\ &\leq C(u)[|||\mathrm{d}_t(u_1 - U_1)^0|||_{0,\Omega_f} + |||\mathrm{d}_t(u_2 - U_2, u_3 - U_3)^0|||_{0,\Omega_p} \\ &\quad + \|(u - U)^0\|_V + \|(u - U)^1\|_V + (\Delta t)^2 + h]. \end{aligned}$$

5. Derivation of the absorbing boundary conditions

In this section we shall derive the absorbing boundary condition (2.8.ii) for the artificial boundary Γ_2 of Ω_p. In this derivation we shall use some results of [9] as well as some ideas given in [11] for obtaining absorbing boundary conditions for anisotropic elastic solids.

Let us consider a wave front arriving normally to Γ_2 with velocity c. Following [8], the strain tensor $\varepsilon(u_2^c)$ on Γ_2 can be written in the form

$$\left.\begin{aligned} &\varepsilon_{rr}(u_2^c) = -\frac{1}{c}\frac{\partial u_{2r}^c}{\partial t}\nu_{pr}, \qquad \varepsilon_{\theta\theta}(u_2) = \frac{u_{2r}^c}{r} = 0, \qquad \varepsilon_{zz}(u_2^c) = -\frac{1}{c}\frac{\partial u_{2z}^c}{\partial t}\nu_{pz}, \\ &\varepsilon_{rz}(u_2^c) = -\frac{1}{2c}\left(\frac{\partial u_{2r}^c}{\partial t}\nu_{pz} + \frac{\partial u_{2z}^c}{\partial t}\nu_{pr}\right), \qquad (r, \theta, z) \in \Gamma_2,\ t \in J. \end{aligned}\right\} \tag{5.1}$$

In particular,

$$\nabla \cdot u_2^c = -\frac{1}{c}\frac{\partial u_2^c}{\partial t} \cdot \nu_p. \tag{5.2}$$

Also,

$$\nabla \cdot u_3^c = -\frac{1}{c}\frac{\partial u_3^c}{\partial t} \cdot \nu_p. \tag{5.3}$$

Next, let us introduce the variables

$$\alpha_1^c = \frac{1}{c}\frac{\partial u_2^c}{\partial t} \cdot \nu_p, \qquad \alpha_2^c = \frac{1}{c}\frac{\partial u_2^c}{\partial t} \cdot \chi_p^1,$$

$$\alpha_3^c = \frac{1}{c}\frac{\partial u_2^c}{\partial t} \cdot \chi_p^2, \qquad \alpha_4^c = \frac{1}{c}\frac{\partial u_3^c}{\partial t} \cdot \nu_p,$$

and set

$$\alpha^c = (\alpha_1^c, \alpha_2^c, \alpha_3^c, \alpha_4^c)^{\mathrm{T}}.$$

Combining the stress–strain relations (2.1) with (5.1), (5.2), and (5.3) shows that the strain-energy density W_p on Γ_2 can be written as a quadratic function $\Pi(\alpha^c) = W_p(\varepsilon(\alpha^c), \nabla \cdot u_3(\alpha^c))$ in the form

$$\Pi(\alpha^c) = \tfrac{1}{2}(\alpha^c)^{\mathrm{T}}\bar{E}_p\alpha^c, \tag{5.4}$$

where $\bar{E}_p \in \mathbb{R}^{4\times 4}$ is the symmetric, positive-definite matrix given by

$$\bar{E}_p = \begin{pmatrix} A+2N & 0 & 0 & Q \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ Q & 0 & 0 & H \end{pmatrix}.$$

Next, note that the momentum equations on Γ_2 are given by

$$c\mathscr{A}\frac{\partial(u_2^c, u_3^c)}{\partial t} = \left(-\sum_{i,j=r,\theta,z}\frac{\partial W_p}{\partial \varepsilon_{ij}}\nu_{pj}, \frac{\partial W_p}{\partial(-\nabla \cdot u_3)}\nu_p\right)$$

for $(r, \theta, z) \in \Gamma_2$, $t \in J$. Alternatively, they can be written in the form (cf. (2.4))

$$\left.\begin{aligned} &\text{(i)} \qquad c[\rho\, \partial u_2^c/\partial t + \rho_f\, \partial u_3^c/\partial t] = -\tau v_p, \\ &\text{(ii)} \qquad c[\rho_f\, \partial u_2^c/\partial t + g\, \partial u_3^c/\partial t] = p v_p,\ (r, \theta, z) \in \Gamma_2,\ t \in J. \end{aligned}\right\} \tag{5.5}$$

Now, we shall write equations (5.5) in terms of the new variables α_i^c, $1 \le i \le 4$. First, note that taking the inner product of (5.5.ii) with the tangent vectors χ_p^m, $m = 1, 2$, gives the relations

$$\frac{\partial u_3^c}{\partial t} \cdot \chi_p^m = -g^{-1} \rho_f \frac{\partial u_2^c}{\partial t} \cdot \chi_p^m, \qquad m = 1, 2.$$

Let $\mathscr{F} = (\tau v_p \cdot v_p, \tau v_p \cdot \chi_p^1, \tau v_p \cdot \chi_p^2, -p)^{\mathrm{T}}$. Then, take the inner product of (5.5.i) with v_p and χ_p^m, $m = 1, 2$, and of (5.5.ii) with v_p to get the equations

$$c^2 \bar{\mathscr{A}} \alpha^c = -\mathscr{F} = \frac{\partial \Pi}{\partial \alpha^c} = \bar{E}_p \alpha^c, \tag{5.6}$$

$\bar{\mathscr{A}} \in \mathbb{R}^{4\times4}$ being the symmetric, positive-definite matrix defined by

$$\bar{\mathscr{A}} = \begin{pmatrix} \rho & 0 & 0 & \rho_f \\ 0 & \rho - g^{-1}(\rho_f)^2 & 0 & 0 \\ 0 & 0 & \rho - g^{-1}(\rho_f)^2 & 0 \\ \rho_f & 0 & 0 & g \end{pmatrix}.$$

Let $S = \bar{\mathscr{A}}^{-\frac{1}{2}} \bar{E}_p \bar{\mathscr{A}}^{-\frac{1}{2}}$, $\bar{\alpha}^c = \bar{\mathscr{A}}^{\frac{1}{2}} \alpha^c$. Then equation (5.6) becomes

$$S \bar{\alpha}^c = c^2 \bar{\alpha}^c. \tag{5.7}$$

Also, in terms of $\bar{\alpha}^c$ the strain-energy density on Γ_2 can be written in the form

$$\bar{\pi}(\bar{\alpha}^c) \equiv \pi(\alpha^c) = \tfrac{1}{2}(\bar{\alpha}^c)^{\mathrm{T}} S \bar{\alpha}^c. \tag{5.8}$$

Let c_i, $1 \le i \le 4$, be the four positive wave velocities satisfying (5.7); that is, solutions of the equation

$$\det(S - c^2 I) = 0.$$

Two of these roots are

$$c_2 = c_3 = \left(\frac{N}{\rho - g^{-1}(\rho_f)^2}\right)^{\frac{1}{2}}$$

and they correspond to the shear modes of propagation. The other two velocities c_1 and c_4 are distinct and they correspond to the compressional modes of propagation and have a more complicated expression in terms of the mass and stiffness coefficients of Ω_p. It can be easily checked that these values coincide with the corresponding ones obtained in [1] by Biot using a different argument.

Let M_i, $1 \le i \le 4$, be the set of orthonormal eigenvectors corresponding to c_i, $1 \le i \le 4$, and let M be the matrix containing the eigenvectors M_i of S as rows, and D be the diagonal matrix containing the eigenvalues c_i^2, $1 \le i \le 4$, of S, so that $S = M^{\mathrm{T}} D M$.

Let

$$\alpha = \left(\frac{\partial u_2}{\partial t}\cdot v_p, \frac{\partial u_2}{\partial t}\cdot \chi_p^1, \frac{\partial u_2}{\partial t}\cdot \chi_p^2, \frac{\partial u_3}{\partial t}\cdot v_p\right)^{\mathrm{T}}$$

be a general velocity on the surface Γ_2 due to the simultaneous normal arrival of waves of velocities c_i, $1 \leq i \leq 4$. Since the M_i are orthonormal, we can write $\bar{\alpha} = \bar{\mathscr{A}}^{\frac{1}{2}}\alpha$ in the form

$$\bar{\alpha} = \sum_{i=1}^{4} [M_i, \bar{\mathscr{A}}^{\frac{1}{2}}\alpha]_e M_i.$$

Let

$$\bar{\alpha}^{c_i} = \bar{\mathscr{A}}^{\frac{1}{2}}\alpha^{c_i} = \frac{1}{c_i}[M_i, \bar{\mathscr{A}}^{\frac{1}{2}}\alpha]_e M_i, \qquad 1 \leq i \leq 4. \tag{5.9}$$

Then, $\bar{\alpha}^{c_i}$ satisfies

$$S\bar{\alpha}^{c_i} = c_i^2 \bar{\alpha}^{c_i}, \tag{5.10}$$

and

$$\bar{\pi}(\bar{\alpha}^{c_i}) = \tfrac{1}{2}(\bar{\alpha}^{c_i})^{\mathrm{T}} S\bar{\alpha}^{c_i}. \tag{5.11}$$

Using (5.6) and (5.11), we see that the force $\mathscr{F}_i$ on Γ_2 corresponding to $\bar{\alpha}^{c_i}$ satisfies the relations

$$\bar{\mathscr{A}}^{\frac{1}{2}}\frac{\partial \bar{\pi}}{\partial \bar{\alpha}^{c_i}} = \bar{\mathscr{A}}^{\frac{1}{2}} S\bar{\alpha}^{c_i} = \bar{E}_p \alpha^{c_i} = -\mathscr{F}_i. \tag{5.12}$$

We observe that the interaction energy among the different types of waves arriving normally to Γ_2 is small compared to the total energy involved [5]. Thus, neglecting such interaction, we can write the total strain energy density

$$\bar{\pi} = \bar{\pi}(\bar{\alpha}) = \sum_{i=1}^{4} \bar{\pi}(\bar{\alpha}^{c_i})$$

as the sum of the partial energies and the total force $\mathscr{F}$ on Γ_2 as the sum of the forces corresponding to each $\bar{\alpha}^{c_i}$, so that, according to (5.12),

$$\mathscr{F} = \sum_{i=1}^{4} \mathscr{F}_i = -\bar{\mathscr{A}}^{\frac{1}{2}} \sum_{i=1}^{4} S\bar{\alpha}^{c_i}.$$

Since we can write $\bar{\mathscr{A}}^{-\frac{1}{2}}\mathscr{F}$ in the form

$$\bar{\mathscr{A}}^{-\frac{1}{2}}\mathscr{F} = \sum_{i=1}^{4} [M_i, \bar{\mathscr{A}}^{-\frac{1}{2}}\mathscr{F}]_e M_i,$$

it follows that

$$S\bar{\alpha}^{c_i} = -[M_i, \bar{\mathscr{A}}^{-\frac{1}{2}}\mathscr{F}]_e M_i, \qquad 1 \leq i \leq 4. \tag{5.13}$$

Thus, combining (5.9), (5.10) and (5.13), we see that

$$c_i^2 \bar{\alpha}^{c_i} = c_i [M_i, \bar{\mathscr{A}}^{\frac{1}{2}}\alpha]_e M_i = S\bar{\alpha}^{c_i} = -[M_i, \bar{\mathscr{A}}^{-\frac{1}{2}}\mathscr{F}]_e M_i, \qquad 1 \leq i \leq 4,$$

so that

$$c_i[M_i, \tilde{\mathscr{A}}^{\frac{1}{2}}\alpha]_e = -[M, \tilde{\mathscr{A}}^{-\frac{1}{2}}\mathscr{F}]_e, \qquad 1 \leq i \leq 4.$$

In matrix form the equation above becomes

$$-M\tilde{\mathscr{A}}^{-\frac{1}{2}}\mathscr{F} = D^{\frac{1}{2}}M\tilde{\mathscr{A}}^{\frac{1}{2}}\alpha,$$

so that after multiplying by $\tilde{\mathscr{A}}^{\frac{1}{2}}M^{\mathrm{T}}$ we obtain the relations

$$-\mathscr{F} = \tilde{\mathscr{A}}^{\frac{1}{2}}S^{\frac{1}{2}}\tilde{\mathscr{A}}^{\frac{1}{2}}\alpha = [(\tilde{\mathscr{A}}^{-1}\tilde{E}_p)^{\mathrm{T}}]^{\frac{1}{2}}\tilde{\mathscr{A}}\alpha = B\alpha.$$

These are the equations used as boundary conditions for the artificial boundary Γ_2. Note that the matrix B is symmetric and positive definite.

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